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THE

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QUARTERLY JOURNAL

OF

PURE AND APPLIED MATHEMATICS.

MULTIPLE POINTS ON SURFACES.

By A. B. Basset, M.A., F.R.S.

1. In a former communication I expressed the opinion that the conic node and the binode are the only *simple* point singularities that an algebraic surface can possess; and that every other singularity is a compound one which, so far as its point constituents are concerned, is formed by the union of two or more conic nodes or binodes. I also pointed out that Segre has shown that every multiple point of order p, at which there is an anautotomic tangent cone, reduces the class of the surface by $p(p-1)^2$; and I gave reasons for thinking that the point constituents of such a multiple point are half the above number of conic nodes. I have not succeeded in obtaining a rigorous proof of this theorem, but I shall assume its truth; and on this hypothesis shall proceed to investigate the point constituents of a multiple point, the tangent cone at which is autotomic or improper.

The investigation of the point constituents of any multiple point involves the solution of two distinct problems. In the first place, the class m of the surface is determined by the equation

$$m = n(n-1)^2 - 2C - 3B \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where C and B are the number of constituent conic nodes and binodes.

In the second place the above number is given by an equation of the form

$$C + B = \lambda \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

accordingly, when the numerical values of m and λ have been found, the solution of the problem is known

When the tangent cone at a multiple point of order p is anautotomic, $\lambda = \frac{1}{2}p(p-1)^2$; and Segre's theorem, which

is explained* in *S.S.* § 13, enables us to ascertain whether any change in the character of a multiple point is produced by the conversion of conic nodes into binodes, or by the union of additional conic nodes or binodes with the multiple point. The value of λ can therefore be usually obtained without much trouble; but the principal difficulty is to determine the class of the surface.

The most direct way of obtaining the class of a surface is to eliminate two of the coordinates between the equations of the surface and its first polars with respect to two arbitrary points; for the eliminant is a binary quantic of the remaining two coordinates, whose degree is equal to the class of the surface. But since the elimination of two variables between three quarternary quantics is an operation of considerable difficulty, recourse will be had to various indirect methods which will be explained hereafter. When the values of the coefficients are not material, I shall frequently denote a quantic by the symbol $(\alpha, \beta, \gamma, \ldots)^n$.

The First Method.

2. The first method, which I shall explain, for finding the reduction of class produced by a multiple point of order p is of a simple character. Let us first consider the surfaces

$$\alpha u_p + u_{p+1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

and

$$\alpha^2 u_p + \alpha u_{p+1} + u_{p+2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

In both surfaces the lines of closest contact are the common generators of the cones u_p and u_{p+1}; but in the first surface these lines lie wholly upon it, whilst in the second they have $(p+2)$-tactic contact with it at A and consequently intersect the surface nowhere else. This shows that a line of closest contact may be regarded as a line which has $(p+2)$-tactic contact with the surface at the multiple point.

The first polars of (3) with respect to C and D, which may be any arbitrary points, are

$$\alpha u'_p + u'_{p+1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

$$\alpha u''_p + u''_{p+1} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

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* References to my former paper "On the Singularities of Surfaces," Vol. XXXVIII., pp. 63 and 159, will be denoted by the letters *S.S.*; whilst similar references to my paper on the "Compound Singularities of Curves," Vol. XXXVII., p. 313, will be denoted by the letters *S.C.*

where the single and double accents denote differentations with respect to γ and δ respectively; whence, eliminating α between (3), (5), and (6), we obtain

$$u_p u'_{p+1} = u_{p+1} u'_p \ldots\ldots\ldots\ldots\ldots\ldots (7),$$

$$u_p u''_{p+1} = u_{p+1} u''_p \ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Equations (7) and (8) represent two cones of degree $2p$, and their $4p^2$ common generators intersect the surface (3) at the points where it is intersected by (5) and (6); but these generators include the $p(p+1)$ lines of closest contact, which do not give rise to ordinary points of intersection. Hence there are only

$$4p^2 - p(p+1) = 3p^2 - p$$

ordinary points of intersection of (3) and its first polars with respect to two arbitrary points; accordingly,

$$m = 3p^2 - p = (p+1)p^2 - p(p-1)^2,$$

showing that the reduction of class produced by a multiple point of order p, the tangent cone at which is anautotomic, is $p(p-1)^2$.

3. We shall now prove a more general theorem:—

When the nodal cone at a multiple point of order p has δ nodal and κ cuspidal generators, all of which are distinct, the reduction of class is

$$p(p-1)^2 + \delta + 2\kappa,$$

and the point constituents of the singularity are

$$C = \tfrac{1}{2}p(p-1)^2 - \delta - 2\kappa, \quad B = \delta + 2\kappa.$$

If we temporarily regard the cones u_p, u'_p, u''_p as curves lying in the plane BCD, the last two will be the first polars of u_p with respect to the points C and D; accordingly, it follows from the theory of plane curves, that if AB is a singular generator of u_p, this line must be repeated one or more times on the cones u'_p and u''_p. If, therefore, AB is repeated s times on these cones, and s or more times on the cone u_p, the number of ordinary common generators will be diminished by s, and the total reduction of the class of the surface will be $p(p-1)^2 + s$.

(i) Let AB be a nodal generator on u_p; then AB is a double generator of u_p and an ordinary generator of u'_p and u''_p; hence $s = 1$.

(ii) Let AB be a cuspidal generator on u_p; then since the first polar of every point has tritactic contact with a curve at a cusp, and bitactic contact with the first polar of any other point, $s=2$.

Combining these results we obtain the preceding expression for the reduction of class; also since the singular generator is not produced by the union of any additional double points with the multiple point, $C+B=\frac{1}{2}p(p-1)^2$, which proves the second part of the theorem.

4. In a former paper,* I have drawn a distinction between *multiple* points and *singular* points on curves; and I have defined the latter as follows: "If an arbitrary straight line drawn through a point P, which is not a multiple point of order r, intersect the curve in r coincident points at P, then P will be called a singular point of order r." We shall now prove that:—

If the nodal cone at a multiple point of order p possesses a singular generator of order 2, *whose constituents are δ nodal and κ cuspidal generators which move up to coincidence along a continuous curve, the total reduction of class is*

$$p(p-1)^2+2\delta+3\kappa-1.$$

It will be sufficient to verify this theorem for the four singular generators of order two, which a quartic cone can possess. The object of the investigation is to find the value of s, which is equal to the order of contact at B of the first polars of C and D, when B is a tacnode, a rhamphoid cusp, an oscnode and a tacnode cusp on a quartic curve.

(i) Let B be a tacnode; then the equation of the quartic curve referred to the triangle BCD is

$$\beta^2v_1^2+2\beta v_1v_2+2v_4=0.....(9),$$

and we may, without loss of generality, write $v_1=\gamma+\delta$. The first polars of C and D are

$$\beta^2v_1+\beta(v_2+v_1v'_2)+v'_4=0.....(10),$$

$$\beta^2v_1+\beta(v_2+v_1v''_2)+v''_4=0.....(11),$$

and the eliminant of (10) and (11) is

$$(v'_2-v''_2)\{v_2(v'_4-v''_4)+v_1(v'_4v''_2-v''_4v'_2)\}=(v'_4-v''_4)^2...(12).$$

* "Compound Singularities of Quintic Curves," *Quart. Jour.*, Vol. xxxvi., p. 363.

Equation (12) is a binary sextic, which shows that (10) and (11) intersect one another in three coincident points at B; hence $s=3$.

(ii) Let B be a rhamphoid cusp; then the equation of the quartic is

$$(\beta v_1+v_2)^2+2v_1V_3=0 \quad \ldots\ldots\ldots\ldots\ldots (13),$$

whence

$$v_4=\tfrac{1}{2}v_2^2+v_1V_3.$$

Substituting this value of v_4 in (12), it will be found that the term which is independent of v_1 vanishes; hence v_1 is a factor of (12), and the polars intersect one another in four coincident points at B; accordingly $s=4$.

(iii) Let B be an osculode; then the equation of the quartic is

$$(\beta v_1+V_2)^2+2v_1^3(k\beta+V_1)=0 \quad \ldots\ldots\ldots\ldots\ldots (14),$$

so that

$$v_2=V_2+kv_1^2,$$

$$v_4=\tfrac{1}{2}V_2^2+v_1^3V_1.$$

Substituting in (12) it will be found that the term involving v_1 vanishes; hence $s=5$.

(iv) Let B be a tacnode cusp; then $k=0$, and we shall find that $s=6$.

Since the constituents of these four singular lines are $\delta=2$, $\kappa=0$; $\delta=1$, $\kappa=1$; $\delta=3$, $\kappa=0$; and $\delta=2$, $\kappa=1$ respectively, the theorem is verified for the special generators considered; and its truth as a general theorem is sufficiently obvious. Also, as long as none of the generators of the cone u_{p+1} pass through AB, the reduction of class is produced by the conversion of conic nodes into binodes; hence

$$C+B=\tfrac{1}{2}p(p-1)^2,$$

accordingly the constituents of the multiple point are

$$C=\tfrac{1}{2}p(p-1)^2-2\delta-3\kappa+1, \quad B=2\delta+3\kappa-1.$$

5. *If the nodal cone at a multiple point of order p possesses a multiple generator of order q, such that r of the tangent planes are coincident, the constituents of the singularity are*

$$C=\tfrac{1}{2}p(p-1)^2-(q-1)^2-r+1, \quad B=(q-1)^2+r-1.$$

(i) Let all the tangent planes along the multiple generator AB be distinct; then since a multiple point of order q on

a curve gives rise to a multiple point of order $q-1$ on the first polar, it follows that the first polars of two arbitrary points intersect in $(q-1)^2$ coincident points at B; hence $s=(q-1)^2$.

(ii) Let r of the tangent planes along AB coincide; then AB is a multiple point of order $q-1$ on the first polar, having $r-1$ coincident tangent planes; accordingly AB will be repeated $r-1$ additional times on the first polars of two arbitrary points, so that $s=(q-1)^2+r-1$. This gives

$$2C+3B=p(p-1)^2+(q-1)^2+r-1,$$

also $$C+B=\tfrac{1}{2}p(p-1)^2,$$

which proves the theorem.

It does not appear to make any difference whether the preceding compound singularities occur on a proper or an improper cone. Putting $q=3$, $r=1$, it follows that the additional reduction of class produced by a triple generator on the nodal cone is 4; and it will hereafter be shown, by an independent investigation, that the additional reduction of class produced by a triple generator when the nodal cone is a quartic cone is the same, whether the cone is (i) a proper one, (ii) a nodal cubic cone and a plane through the nodal generator, (iii) two planes and a quadric cone passing through their lines of intersection.

6. We have shown in § 3 that a nodal generator AB on the nodal cone u_p produces an additional reduction of class equal to 1, and converts a conic node into a binode; but it follows from *S.S.* § 13, that if AB is also an ordinary generator of the cone u_{p+1}, an additional reduction of class equal to 2 is produced, and an additional conic node is added to the constituents of the multiple point. We shall now show that:—

(i) *If AB is a singular generator of order 2 on the nodal cone u_p, which produces a reduction of class equal to s; then when AB is an ordinary generator of the cone u_{p+1}, the constituents of the multiple point are*

$$C=\tfrac{1}{2}p(p-1)^2-s+2, \quad B=s-1.$$

It will be sufficient to establish the theorem for a cuspidal generator, since the method of proof is the same for any other singular generator. Consider the surface

$$\alpha^{n-p}(\beta^{p-2}v_1^2+\beta^{p-3}v_3+\ldots)+\alpha^{n-p-1}(\beta^{p+1}w_0+\beta^p w_1+\ldots)$$
$$+\alpha^{n-p-2}u_{p+2}+\ldots u_n=0.$$

By § 3, the constituents of the singularity are given by the equations

$$2C+3B=p(p-1)^2+2,$$

$$C+B=\tfrac{1}{2}p(p-1)^2,$$

also AB has $(p+1)$-tactic contact with the surface at A, and $(p+2)$-tactic contact with the first polar of any point; but, if $w_0=0$, AB has $(p+2)$-tactic contact with the surface at A, hence the surface and its first polar intersect at an additional point which lies on AB. Also it can be shown as in *S.S.* § 13, that this reduction of class is produced by the union of an additional double point with the multiple point; hence the constituents of the singularity are given by

$$2C+3B=p(p-1)^2+3,$$

$$C+B=\tfrac{1}{2}p(p-1)^2+1,$$

giving $$C=\tfrac{1}{2}p(p-1)^2, \quad B=1.$$

When the singular line AB is such that the additional reduction of class produced by it is s, it follows by parity of reasoning that when AB is an ordinary generator of u_{p+1}, the equations are

$$2C+3B=p(p-1)^2+s+1,$$

$$C+B=\tfrac{1}{2}p(p-1)^2+1.$$

The following theorem is an extension of this result:—

(ii) *If the nodal cone at a multiple point of order p possesses* δ *nodal generators, which are ordinary generators of the cone* u_{p+1}*; then when all coincide the constituents of the singularity are*

$$C=\tfrac{1}{2}p(p-1)^2-\delta+2, \quad B=2\delta-2.$$

To prove the theorem it will be sufficient to consider the case of a tacnodal generator on a quartic nodal cone.

When the cone u_{p+1} passes through two distinct nodal generators AB and AB' on the cone u_p, the effect is to add two conic nodes to the constituents of the singularity, so that in the case of a quartic node $2C+3B=36+2+2=40$, and we shall show that when AB and AB' coincide, the effect is to produce a further reduction of class equal to 2. Consider the surface

$$\alpha(\beta^2v_1^2+2\beta v_1v_2+2v_3)+k\beta^4v_1+\beta^3w_2+\ldots w_5=0\ldots(15),$$

where $v_1 = \gamma + \delta$ and k is a constant. Equation (7) now become

$$(\beta^2 v_1^2 + 2\beta v_1 v_2 + 2v_4)\,(k\beta^4 + \beta^3 w_2' + \ldots w_5')$$

$$= 2\,(k\beta^4 v_1 + \beta^3 w_2 + \ldots w_5)\,\{\beta^2 v_1 + \beta\,(v_2 + v_1 v_2') + v_4'\} \ldots (16),$$

or

$$k\beta^6 v_1^2 + \beta^5 v_1\,(2k v_1 v_2' + 2w_2 - k v_1 w_2') + \ldots = 0 \ldots (17).$$

Now write down the equation corresponding to (8) and subtract, and it will be found that we shall obtain an equation of the form

$$\beta^5 v_1^2 \Omega_1 + \beta^4 v_1 \Omega_2 + \ldots = 0 \ldots \ldots\ldots (17A),$$

from which it is not difficult to show that (17) and (17A) intersect in 54 ordinary generators, and in 10 coincident generators along AB. But, since AB is four times repeated amongst the lines of closest contact of (15), the total number of ordinary lines of intersection of (17) and (17A) are

$$64 - 10 - (20 - 4) = 38.$$

Whence $$m = 38 = 80 - 2C - 3B,$$

giving $$2C + 3B = 42 = 36 + 6.$$

From this it follows that the coincidence of the two generators AB, AB' produces a further reduction of class equal to 2; and by taking a third generator AB'', which is a nodal generator on u_p and an ordinary generator on u_{p+1}, and making it coincide with AB, it can be shown that the additional reduction of class is $2 + 2 = 4$. Generalizing, it follows that when there are δ coincident nodal generators

$$2C + 3B = p\,(p-1)^2 + 4\delta - 2,$$

also $$C + B = \tfrac{1}{2}\,p\,(p-1)^2 + \delta,$$

which proves the theorem.

Multiple Points in which the Cone consists of Planes intersecting in the same Straight Line.

7. We shall now discuss multiple points in which the cone degrades into p planes intersecting in the same straight line, and shall commence with the following theorem.

When a multiple point of order p consists of p distinct planes intersecting in a point, the reduction of class is

$$\tfrac{1}{2}\,p\,(p-1)\,(2p-1),$$

and the point constituents of the singularity are

$$C=\tfrac{1}{2}p(p-1)(p-2),\quad B=\tfrac{1}{2}p(p-1).$$

But when the planes intersect in the same straight line, the reduction of class is $(p+1)(p-1)^2$, *and the point constituents of the singularity are*

$$C=\tfrac{1}{2}(p-1)^2(p-2),\quad B=(p-1)^2.$$

By means of the theorem *S.S.* § 13, it can be shown in both cases that the reduction of class does not arise from the union of any additional conic nodes or binodes with the multiple point; hence the reduction is caused by the conversion of conic nodes into binodes.

In the first case, when the planes intersect in the same point, the number of their lines of intersection is $\frac{1}{2}p(p-1)$; hence

$$2C+3B=p(p-1)^2+\tfrac{1}{2}p(p-1),$$

also
$$C+B=\tfrac{1}{2}p(p-1)^2,$$

whence
$$C=\tfrac{1}{2}p(p-1)(p-2),\quad B=\tfrac{1}{2}p(p-1)\ldots(18).$$

To prove the second case, we may proceed as in *S.S.* § 23, and employ a surface of degree $p+1$, which is

$$\alpha v_p+u_{p+1}=0\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

where
$$u_{p+1}=\beta^{p+1}w_0+\beta^p w_1+\ldots w_{p+1}.$$

The first polars of C and D are

$$\alpha v'_p+u'_{p+1}=0,\quad \alpha v''_p+u''_{p+1}=0\ldots\ldots\ldots(20).$$

Multiplying the first of (20) by γ and the second by δ and adding, we obtain

$$p\alpha v_p+\beta^p w_1+2\beta^{p-1}w_2+\ldots(p+1)w_{p+1}=0\ldots..(21).$$

Eliminating α between (19) and (21), we obtain

$$p\beta^{p+1}w_0+(p-1)\beta^p w_1+\ldots-w_{p+1}=0\ldots\ldots..(22).$$

Eliminating α between (20), we obtain

$$(\beta^p w_1'+\beta^{p-1}w_2'+\ldots)v_p''=(\beta^p w_1''+\beta^{p-1}w_2''+\ldots)v_p'\ldots(23).$$

Equations (22) and (23) represent two cones of degrees $p+1$ and $2p-1$, which possess $(p+1)(2p-1)$ common generators; and this number is equal to the number of ordinary points of intersection of (19) and the first polars of

two arbitrary points. Hence

$$m = (p+1)(2p-1) = (p+1)p^2 - 2C - 3B,$$

accordingly $$2C + 3B = (p+1)(p-1)^2 \ldots\ldots\ldots\ldots (24),$$

also $$C + B = \tfrac{1}{2}p(p-1)^2,$$

whence $$C = \tfrac{1}{2}(p-1)^2(p-2), \quad B = (p-1)^2 \ldots\ldots (25).$$

8. *When s tangent planes coincide, the reduction of class is*

$$(p+1)\{(p-1)^2 + s - 1\}.$$

In this case $v_p = \delta^s v_{p-s}$; hence

$$v_p' = \delta^s v'_{p-s}, \quad v_p'' = \delta^{s-1}(\delta v''_{p-s} + s v_{p-s});$$

accordingly (23) contains δ^{s-1} as a factor which must be rejected, and the resulting cone is of degree $2p-s$. Whence

$$m = (p+1)(2p-s) = (p+1)p^2 - 2C - 3B,$$

giving $$2C + 3B = (p+1)\{(p-1)^2 + s - 1\} \ldots\ldots (26),$$

from which it follows that each successive coincident plane produces an additional reduction of class equal to $p+1$. When all the planes coincide, $p = s$, and the reduction becomes $p(p^2-1)$, a result which is known to be true when $p = 2$ and $p = 3$.

9. We shall now explain a method for determining the number of constituent conic nodes and binodes, when some of the planes coincide, which depends upon the theorem of *S.S.* § 13.

Two cones of degree n which have a common vertex possess n^2 common generators; and a pair of such cones may be regarded as an improper cone of degree $2n$ which has n^2 nodal generators. If the two cones have an additional common generator they must coincide; hence a cone of degree n twice repeated may be regarded as an improper cone of degree $2n$ which has n^2+1 nodal generators. From this it follows that a pair of coincident planes may be regarded as a *binodal* quadric cone, the positions of whose nodal generators are indeterminate. Similarly three coincident planes may be regarded as an improper cubic cone having $2+2+2=6$ common generators; and generally if t_s be the number of nodal generators when there are s coincident planes,

$$t_s = t_{s-1} + 2(s-1),$$

the solution of which is $t_s = s(s-1)$. Accordingly s coincident planes may be regarded as a cone of degree s which has $s(s-1)$ nodal generators, the positions of which are indeterminate.

10. *When s coincident planes coincide, the constituents of the singularity are**

$$C = \tfrac{1}{2}(p-1)^2(p-2) - (p+1)(s-4),$$

$$B = (p-1)^2 + (p+1)(s-3); \text{ when } p+1 \not< s(s-1),$$

and

$$C = \tfrac{1}{2}(p-1)^2(p-2) + (s-1)(3s-p-1),$$

$$B = (p-1)^2 - (s-1)(2s-p-1); \quad \text{when } p+1 \not> s(s-1).$$

The equation of the surface is

$$\alpha^{n-p}\delta^s v_{p-s} + \alpha^{n-p-1}u_{p+1} + \ldots u_n = 0 \ldots\ldots\ldots\ldots (27).$$

In the first case, each of the $p+1$ lines of intersection of the cone u_{p+1} with the plane δ may be regarded as nodal generators of the cone δ^s; hence the number of additional nodes is $p+1$, and

$$C + B = \tfrac{1}{2}p(p-1)^2 + p + 1 \ldots\ldots\ldots\ldots (28).$$

Combining this with (26) we obtain the first result. But in the second case there are only $s(s-1)$ additional nodes, whence

$$C + B = \tfrac{1}{2}p(p-1)^2 + s(s-1) \ldots\ldots\ldots\ldots (29),$$

which by virtue of (26) gives the second result.

When $p = s$, $p+1 < p(p-1)$ except when $p = 1$ and $p = 2$; and in the latter case the singularity is an ordinary unode, and the second result gives $C = 3$, $B = 0$, which is right. But when $p = s > 2$, the proper formulæ are the first ones, and we obtain

$$C = \tfrac{1}{2}(p-1)^2(p-2) - (p+1)(p-4), \quad B = (p-1)^2 + (p+1)(p-3) \ldots (30),$$

where $p > 2$. For a cubic node of the fifth kind, $p = 3$, $s = 2$, so that the second formulæ are the proper ones, which give $C = 4$, $B = 4$; but for a cubic node of the sixth kind $p = s = 3$, so that we must use (30) which give $C = 6$, $B = 4$.

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* The symbols $<$ and $>$ for "less than" and "greater than" are well known; but there does not appear to be any established notation for the phrases "not less than" and "not greater than." I shall therefore employ the symbols $\not<$ and $\not>$ for these phrases, and venture to recommend their use to mathematicians.

Tangents drawn from a Multiple or Singular Point to a Curve.

11. Before explaining the second method it will be necessary to consider a few preliminary theorems connected with the number of tangents which can be drawn to a *plane* curve from a multiple or singular point on it.

(i) *Let r tangents at a multiple point of order p coincide; then, if t be the number of tangents which can be drawn from the point, and m the class of the curve,*

$$t = m - 2p + r - 1,$$

The equation of the curve may be taken to be

$$\alpha^2\gamma^r u_{p-r} + 2\alpha u_{p+1} + u_{p+2} = 0 \ldots\ldots\ldots\ldots\ldots (31),$$

and the equation of the tangents which can be drawn from A to the curve is

$$u^2_{p+1} = \gamma^r u_{p-r} u_{p+2};$$

whence

$$t = 2p + 2 \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

Also $\quad \delta = \frac{1}{2}p(p-1) - r + 1, \quad \kappa = r - 1,$

accordingly

$$m = (p+2)(p+1) - p(p-1) + 2(r-1) - 3(r-1)$$
$$= 4p - r + 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33);$$

whence, by (32),

$$t = m - 2p + r - 1 \ldots\ldots\ldots\ldots\ldots (34).$$

When all the tangents are distinct $r = 1$, and

$$t = m - 2p \ldots\ldots\ldots\ldots\ldots\ldots (34A).$$

(ii) *If the point possesses s pairs of tacnodal branches, r of which are coincident,*

$$t = m - 2p + 2r - 2.$$

Let the equation of the curve be

$$\alpha^2\gamma^{2r} u^2_{s,r} u_{p-2s} + 2\alpha\gamma^r u_{s-r} u_{p-s+1} + u_{p+2} = 0,$$

then, by *S. C.* § 5,

$$\delta = \tfrac{1}{2}p(p-1) + s - 2r + 2, \quad \kappa = 2r - 2;$$

whence

$$m = 4p - 2s - 2r + 4,$$

and

$$t = m - 2p + 2r - 2.$$

When $r=1$, all the pairs of tacnodal branches are distinct, and

$$t=m-2p.$$

(iii) *If the point possesses r pairs of coincident tacnodal branches and σ ordinary branches, which coincide with the tacnodal branches,*

$$t=m-2p+2r+\sigma-2.$$

Let the equation of the curve be

$$\alpha^2\gamma^{2r+\sigma}u_{p-2r-\sigma}+2\alpha\gamma^r u_{p-r+1}+u_{p+2}=0.$$

In *S. C.* § 5, we must put $s=r$; so that, if all the ordinary branches were distinct, we should have $\delta=\frac{1}{2}p\,(p-1)-r+2$; but, by *S. C.* § 4, every *ordinary* branch which coincides with the tacnodal branches changes a node into a cusp; whence

$$\delta=\tfrac{1}{2}p\,(p-1)-r-\sigma+2,\quad \kappa=2r+\sigma-2.$$

Accordingly

$$m=4p-4r-\sigma+4,\quad t=m-2p+2r+\sigma-2.$$

Multiple Points. The Second Method.

12. Let A be a multiple point of order p on a surface; then the tangent cone *at* A will be called the *nodal cone*. With A as vertex a second cone can be described, every generator of which touches the surface at some other point P. This cone will be called the *tangent cone from the node*, and I shall denote it by the letter N. The equation of the cone N can be obtained by writing the equation of the surface in the binary form

$$(u_p,\ u_{p+1},\ \ldots,\ u_n)(\alpha,\ 1)^{n-p}=0\ \ldots\ldots\ldots\ldots(35),$$

and equating its discriminant to zero.

Let D be any point in space; then, with D as vertex, a third cone can be described, every generator of which touches the surface. This cone will be called the *tangent cone from* D, and will be denoted by the letter T.

The class of the cone T is in general equal to that of the surface; but, if D is a conic node, it follows that the class of T is equal to the number of tangent planes which can be drawn to the surface through a line passing through a conic node. Let E be any arbitrary point, D a conic node on the surface; then ED will be a nodal generator of the tangent

cone from E. Hence, by the theory of plane curves, the number of tangent planes which can be drawn through the line ED to the tangent cone from E, and therefore to the surface, is $m-4$, where m is the class of the surface; and, since this is equal to the number of tangent planes which can be drawn through any line through D to the tangent cone *from* D, $m-4$ is the class of the cone T when its vertex is a conic node.

We thus obtain the first of the following theorems, and the remaining four can be proved in a similar manner.

(i) *The number of ordinary tangent planes which can be drawn to a surface through a line passing through a conic node is* $m-4$.

(ii) *When the line passes through two conic nodes the number is* $m-8$.

(iii) *When the line passes through a binode the number is* $m-3$.

(iv) *When the line passes through two binodes, the number is* $m-6$.

(v) *When the line passes through a conic node and a binode the number is* $m-7$.

13. We shall now prove a more general theorem:—

When a surface of degree n has a multiple point of order p, at which there is an anautotomic tangent cone, the degree of the tangent cone from the multiple point is $n(n-1)-p(p+1)$; *and its class is* $m-2p(p-1)$, *where* $m=n(n-1)^2-p(p-1)^2$.

Let A be the multiple point; then any plane section of the surface through A has a multiple point of order p at A; and the degree of the tangent cone *from* A is equal to the number of tangent lines which can be drawn from A to the section. Now the class of the section is $n(n-1)-p(p-1)$; and, by § 11, (i), the number of tangent lines which can be drawn from A to the section is

$$n(n-1)-p(p-1)-2p=n(n-1)-p(p+1).$$

In the next place let D be any point in space, T the tangent cone from D to the surface; then DA is a generator of the cone T, and the class of T is the same as that of the surface. Since the nodal cone *at* A is anautotomic, its class is $p(p-1)$; hence $p(p-1)$ planes can be drawn through DA, which are tangent planes to the nodal cone, and also to the

surface and the cone T. Accordingly DA is a multiple generator on the cone T, through which $p(p-1)$ *distinct* tangent planes can be drawn to the cone; and, by § 11, (i), the number of *ordinary* tangent planes which can be drawn through DA to the cone T, and also the surface, is $m-2p(p-1)$. But every ordinary tangent plane, which can be drawn through DA to the surface, is a tangent plane to the tangent cone *from* the multiple point A and *vice versâ*; hence the class of this cone is $m-2p(p-1)$.

14. It remains to find the value of m. The surface

$$\alpha\{\beta^{p-1}(P\gamma+Q\delta)+\beta^{p-2}v_2+\ldots v_p\}+\beta^p(R\gamma+S\delta)+\ldots w_{p+1}=0 \quad \ldots\ldots(36)$$

has a multiple point of order p at A, and AB is one of the lines of closest contact which, in the case of a multiple point of order p on a surface of degree $p+1$, is a line lying altogether in the surface. Let the plane $\alpha=\lambda\beta$ cut AB in B'; then the equation of the section is

$$\{\lambda(P\gamma+Q\delta)+R\gamma+S\delta\}\beta^p+(\lambda v_2+w_2)\beta^{p-1}+\ldots=0;$$

hence, if $\lambda=-S/Q$, the tangent plane to the surface at B' will pass through AD. Hence one point exists on each line of closest contact, the tangent plane at which passes through AD; accordingly the number of tangent planes which can be drawn through AD to the surface is $p(p+1)$. Whence

$$m-2p(p-1)=p(p+1),$$

which gives $$m=(p+1)p^2-p(p-1)^2.$$

The first term is the class of an anautotomic surface of degree $p+1$, whilst the second term gives the reduction of class produced by a multiple point of order p. We thus obtain another proof of Segre's theorem.

15. We shall now apply this method to prove the theorem of § 3.

If BCD is any plane passing through D, the section of the nodal cone by this plane will be a curve of degree p having δ nodes and κ cusps; hence by the theory of plane curves, the number of tangent lines which can be drawn through D to the section is $p(p-1)-2\delta-3\kappa$, and this is consequently the number of *distinct* tangent planes which can be drawn through DA to the nodal cone. But every plane through DA and

a nodal generator is equivalent to two coincident tangent planes, and every such plane through a cuspidal generator is equivalent to three coincident tangent planes. Hence DA is a singular generator of the tangent cone from D (or the cone $T=0$) of order $p(p-1)$, having δ pairs of coincident tangent planes, corresponding to each nodal generator on the nodal cone, and κ planes consisting of three coincident tangent planes which correspond to each cuspidal generator. From this it follows from § 11, (i), that the number of ordinary tangent planes which can be drawn through DA and the cone T is $m-2p(p-1)+\delta+2\kappa$; and the above expression is equal to the number of tangent planes which can be drawn through DA to the surface.

Now the nodal and cuspidal generators of the nodal cone will not in general coincide with the lines of closest contact; hence, as in § 14, the number of tangent planes which can be drawn through DA to the surface is equal to the number of lines of closet contact, which is $p(p+1)$: accordingly

$$m-2p(p-1)+\delta+2\kappa=p(p+1),$$

or
$$m=(p+1)p^2-p(p-1)^2-\delta-2\kappa,$$

which shows that the reduction of class is $p(p-1)^2+\delta+2\kappa$.

To prove the remaining portion of the theorem, it can easily be shown from *S. S.* § 13, that the reduction of class arises from the conversion of conic nodes into binodes, and not from the union of any additional double points with the multiple point; whence

$$C+B=\tfrac{1}{2}p(p-1)^2 \ldots\ldots\ldots\ldots\ldots\ldots(37),$$

also, since the reduction of class is $2C+3B$, we obtain

$$2C+3B=p(p-1)^2+\delta+2\kappa \ldots\ldots\ldots\ldots(38).$$

Solving (37) and (38), we obtain the required result.

Unodes.

16. The equation of a surface having a unode at A and the plane ABC as the uniplane is

$$\alpha^{n-2}\delta^2+\alpha^{n-3}(\delta^3V_0+\delta^2V_1+\delta V_2+V_3)+\alpha^{n-4}u_4+\ldots u_n=0\ldots.(39),$$

where $V_n=(\beta,\gamma)^n$; from which it follows that the section of the surface by the uniplane is a curve of the n^{th} degree having a triple point at A, the tangents at which are given by the equation $V_3=0$. There are accordingly three primary species

of unodes, according as the triple point is of the first, second or third kinds. Now Salmon has stated *without proof* that, in the case of a *cubic surface*, these three species of unodes respectively reduce the class by 6, 7, and 8; and that their respective constituents are $C=3$, $B=0$; $C=2$, $B=1$,; $C=1$, $B=2$. There is, however, a distinction between a unode on a cubic surface and one in any other surface; for in the former case, the section by the uniplane consists of three straight lines intersecting at the unode; whereas, in the latter case, the three tangents at the triple point do not lie in the surface but have quadritactic contact with it at the unode. We shall therefore enquire whether Salmon's results for cubic surfaces are special ones, or whether they are generally applicable to all surfaces.

Since the characteristics of a unode on a quartic are the same as on any other surface, we shall employ the surface

$$\delta^3(\alpha V_0+W_1)+3\delta^2(\alpha^2+\alpha V_1+W_2)+3\delta(\alpha V_2+W_3)+\alpha V_3+W_4=0 \quad \ldots\ldots\ldots(40),$$

in which A is the unode and ABC the uniplane; and we shall make use of the *second* method explained in § 13.

Writing down the discriminant of (40), regarded as a binary cubic $(\delta,\ 1)^3=0$, it follows that the tangent cone from D is of the 10^{th} degree, and that the term containing the highest power of α is $\alpha^7 V_3$, which shows that DA is a triple generator of the *first* kind on the cone. Accordingly the number of the tangent planes which can be drawn through DA to the surface is $m-6$.

The tangent cone from A is

$$(\delta^3 V_0+3\delta^2 V_1+3\delta V_2+V_3)^2=12\delta^2(\delta^3 W_1+3\delta^2 W_2+3\delta W_3+W_4) \quad \ldots\ldots\ldots(41),$$

which is of the 6^{th} degree and has three nodal generators which are the lines of intersection of the planes $\delta=0$, $V_3=0$; hence this cone is of the 24^{th} class; and therefore 24 tangent planes can be drawn to it through DA. But each of these 24 planes is a tangent plane of the surface; accordingly $m-6=24$, giving $m=30$, which is right. See *S.S.* § 14.

When two of the tangents at the triple point on the section by the uniplane coincide, we may take $V_3=\beta^2\gamma$. In this case AB is a nodal and AC a tacnodal generator of the cone (41), so that its class is still equal to 24. But the line DA now becomes a triple generator of the second kind on the tangent cone from D; hence the number of tangent planes which can

be drawn through it is $m-5$. Accordingly $m-5=24$, giving $m=29$. The coincidence of two of the tangents therefore reduces the class by 7, and changes one of the three constituent conic nodes into a binode.

In the same way it can be shown that when all three tangents at the triple point coincide, the reduction of class is 8 and the constituents of the singularity are $C=1$, $B=2$. We have thus shown that Salmon's results for a cubic surface are of general application.

Binodes.

17. In *S.S.* §§ 13 and 14 I have explained the theory of the binode whose constituents are two conic nodes; and have pointed out that when the surface is a cubic this becomes Salmon's binode B_4. By means of the theory of the Birational Transformation of Surfaces, which I shall not attempt to discuss on the present occasion, it can be shown that the equation of a surface having at A a singularity formed by the union of a conic node and a binode is

$$\alpha^{n-3}(\alpha v_1 + W\beta^2)(\alpha w_1 + W\beta^2) + \alpha^{n-3}(\beta w_2 + w_3)$$

$$+ \alpha^{n-4}(\beta^3 W_1 + \beta^2 W_2 + \beta W_3 + W_4) + \alpha^{n-5}u_5 + \ldots u_n = 0 \ldots (42),$$

where the v's, w's, and W's are binary quantics of (γ, δ); and a similar result can be obtained when the singularity at A is formed by the union of a multiple point of the first kind and a binode. The section of (42) by the plane $v_1 = w_1$ is a curve having a rhamphoid cusp at A; and when $n=3$ the result fails, as might be expected, since the properties of the binode on (42) are different from those of Salmon's binode B_5 *on a cubic surface*, although the constituents of the latter can be shown to be $C=1$, $B=1$.

Salmon's binode B_6 is a peculiar one. I have shown in *S.S.* § 37 that when a *fixed* plane *osculates* a surface along a straight line, there are in general $n-1$ ordinary binodes on the line. Let $n=3$, AB be the line, A one of the binodes upon it, and ABC the fixed osculating plane; then the equation of the cubic surface is

$$\alpha\delta(p\beta + q\gamma + r\delta) + \beta^2\delta + \beta\delta w_1 + w_3 = 0 \ldots\ldots (43).$$

Let the plane $\beta' = \beta - \lambda\alpha = 0$ cut AB in A', and change the tetrahedron of reference to $A'BCD$; then (43) becomes

$$\alpha\delta\{(\lambda\alpha + \beta')p + q\gamma + r\delta\} + (\lambda\alpha + \beta')^2\delta + (\lambda\alpha + \beta')\delta w_1 + w_3 = 0,$$

accordingly, if $\lambda + p = 0$, A' will be the other binode; and, if $p = 0$, the latter will coincide with A. In this case (43) becomes

$$\alpha\delta\,(q\gamma + r\delta) + \beta^2\delta + \beta\delta w_1 + w_2 = 0,$$

and the singularity at A is Salmon's binode B_6.

As Salmon states *without proof* that the constituents of B_6 on a cubic surface are $B = 2$, I have given the foregoing investigation; since, for all that Salmon has proved to the contrary, B_6 might be composed of three conic nodes which move up to coincidence along a continuous curve.

The special features of compound singularities on cubic surfaces are due to the fact that a proper cubic surface, unlike a proper cubic curve, can possess more than one double point; but, since any plane section through the compound singularity must necessarily be an improper cubic curve, in which the corresponding singularity in plane geometry appears in a degraded form, the properties of the compound singularity will be different from what they would be when the degree of the surface is such that the plane section is a proper curve.

Quartic Nodes.

18. I have stated in § 5 that as long as the tangent planes are *distinct*, it does not appear to make any difference to the reduction of class whether a multiple line occurs on a proper or an improper cone. To illustrate this, let us consider the sextic surfaces

$$\alpha^2\,(\delta^2 v_0 + \delta v_1 + \beta\gamma)\,(M\gamma + N\delta)\,\delta$$
$$+ \alpha\,(\delta^4 V_1 + 4\delta^3 V_2 + 6\delta^2 V_3 + 4\delta V_4 + V_5) + \delta^4 W_2 + \ldots W_6 = 0 \ldots (44).$$

and

$$\alpha^2(\delta^3 + \delta^2 v_1 + \delta\gamma w_1 + p\beta\gamma^2)\,\delta + \alpha(\delta^4 V_1 + \ldots) + \delta^4 W_2 + \ldots W_6 = 0 \ldots (45),$$

Both surfaces have conic nodes at D; but in (44) the nodal cone at A consists of a pair of planes and a quadric cone passing through their line of intersection, whilst in (45) it consists of a nodal cubic cone and a plane passing through the nodal generator.

(i) In (44) the plane δ intersects the quadric cone in the lines AB and AC, whilst the plane $M\gamma + N\delta$ intersects the cone in the line AB and another line AE. Hence, according to the preceding theory

$$m = 150 - 36 - 1 - 1 - 4 - 2 = 106.$$

To verify this result, write (44) in the binary form

$$(a, b, c, d, e)(\delta, 1)^4 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (46),$$

then

$$\left.\begin{aligned} a &= Nv_0\alpha^2 + \alpha V_1 + W_2 \\ b &= \tfrac{1}{4}(Mv_0\gamma + Nv_1)\alpha^2 + \alpha V_2 + W_3 \\ c &= \tfrac{1}{6}(Mv_1 + N\beta)\alpha^2\gamma + \alpha V_3 + W_4 \\ d &= \tfrac{1}{4}M\alpha^2\beta\gamma^2 + \alpha V_4 + W_5 \\ e &= \alpha V_5 + W_6 \end{aligned}\right\} \ldots\ldots\ldots (47).$$

The equation of the tangent cone from D is

$$I^3 - 27J^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (48),$$

where

$$\left.\begin{aligned} I &= ae - 4bd + 3c^2 \\ J &= ace + 2bcd - ad^2 - b^2e - c^3 \end{aligned}\right\} \ldots\ldots\ldots (49).$$

Substituting the above values of a, b, c, d, e from (47) in the discriminantal equation (48), it will be found to be of the form

$$\alpha^{12}\gamma^6\beta^2\sigma_1^2\tau_2 + \alpha^{11}\gamma^3\tau_{10} + \alpha^{10}\tau_{14} + \ldots\tau_{24} = 0 \ldots\ldots (50),$$

where $\sigma_1 = 0$ is the equation of the plane DAE, and $\tau_n = (\beta, \gamma)^n$. Hence DA is a multiple line on the cone (50), having three pairs of coincident tacnodal sheets and two pairs of coincident ordinary sheets. Regarding (50) temporarily as a curve, it follows from *S.C.* § 5 that if the ordinary branches (which correspond to the sheets of the cone) were distinct, the constituents of the multiple point at A would be $\delta = 65$, $\kappa = 4$; but the coincidence of the two pairs of ordinary branches converts two nodes into two cusps, whence the constituents of the multiple point are $\delta = 63$, $\kappa = 6$. Hence, if μ be the class* of (50), it can be shown as in § 11 that the number of tangents which can be drawn from A to the cone is $\mu - 18$. Now if m be the class of the surface $\mu = m - 4$, so that the number of tangent planes which can be drawn through the line DA to the cone (50), and therefore to the surface, is $m - 22$.

Each of these $m - 22$ tangent planes is a tangent plane to the tangent cone to the surface *from* A; also this cone is of

* The reason why μ is not equal to $21 \times 23 - 126 - 18 = 408$ is that the cone (50) possesses nodal and cuspidal generators, which arise from a certain number of its generators being double and stationary tangents to the surface.

the 10th degree and AD is the only nodal generator it possesses; hence its class is $90-2=88$, and the number of tangent planes which can be drawn to it, and therefore to the surface, through AD is $88-4=84$. Accordingly $m-22=84$, giving $m=106$, which agrees.

(ii) In (45), AB is the nodal generator of the cubic cone, and the plane δ intersects the cone in AB and AC; hence, according to the preceding theory,

$$m=150-36-1-4-2=107.$$

To verify this, proceed as before and it will be found that the equation of the tangent cone from D is of the form

$$\alpha^{12}\gamma^6\beta^2\tau_4+\alpha^{11}\gamma^3\tau_{10}+\ldots\tau_{21}=0 \ldots\ldots\ldots\ldots (51),$$

from which it can be shown that the number of tangent planes which can be drawn through DA to the surface is $m-23$. Hence, as in (i), we obtain $m-23=84$, giving $m=107$, which agrees.

The following additional cases could be worked out without difficulty: (iii) the plane δ coincides with one of the nodal tangent planes; (iv) the cubic cone is cuspidal, whilst δ is an arbitrary plane through the cuspidal generator; (v) the plane δ is the cuspidal tangent plane.

19. *When the nodal cone consists of a quadric cone and two coincident planes, its point constituents are* $C=12$, $B=8$.

Let the coincident planes cut the quadric cone in AB and AC. Then, from the foregoing theory, we should anticipate that each of the triple lines AB and AC would produce a reduction of class equal to 4; also the union of the two planes gives rise to two additional conic nodes. Hence the total reduction of class ought to be $36+4+4+2+2=48$, giving

$$2C+3B=48,$$

$$C+B=20,$$

from which we obtain $C=12$, $B=8$. But, since the two planes coincide, the lines AB and AC are of the nature of triple lines of the second kind, and we are consequently not at liberty to assume that the reduction of class produced by each of them is equal to 4. I shall therefore give an analytical investigation by means of the sextic surface

$$\alpha^2(\delta^2+\delta v_1+\beta\gamma)\,\delta^2+\alpha\,(\delta^4V_1+4\delta^3V_2+6\delta^2V_3+4\delta V_4+V_5)$$
$$+\delta^4W_2+\ldots W_6=0 \ldots\ldots\ldots\ldots (52).$$

The values of a, b, c, d, e are

$$a = \alpha^2 + \alpha V_1 + W_2,$$
$$b = \tfrac{1}{4}\alpha^2 v_1 + \alpha V_2 + W_3,$$
$$c = \tfrac{1}{6}\alpha^2\beta\gamma + \alpha V_3 + W_4,$$
$$d = \tfrac{1}{4}\alpha V_4 + W_5,$$
$$e = \alpha V_5 + W_6.$$

Writing down the discriminantal equation of the tangent cone from D, it will be found that the highest power of α is α^{11}, and the term involving it is $27c^3e\,(3ac - 2b^2)$; and that its coefficient (rejecting constant factors) is $\beta^3\gamma^3 V_5\,(\beta\gamma - \frac{1}{4}v_1^2)$; also the coefficient of α^{10} does not involve β or γ as a factor. From this it follows that if μ be the class of the cone, the number of tangent planes which can be drawn through DA to the surface is $\mu - 22 = m - 26$, since $\mu = m - 4$.

The tangent cone *from* A has five nodal generators, which are the lines of intersection of the plane δ and the cone V_5; also AD is another nodal generator of the tangent cone; hence its class is $90 - 10 - 2 = 78$. The number of tangent planes which can be drawn to this cone, and therefore to the surface, through AD is accordingly $78 - 4 = 74$; and we thus obtain $m - 26 = 74$, giving $m = 100$.

Let x be the reduction of class produced by each of the lines AB and AD, then

$$m = 100 = 150 - 36 - 2x - 2 - 2 - 2,$$

giving $$x = 4,$$

which verifies the correctness of our conclusions.

20. *When the nodal cone at a quartic node consists of a quadric cone twice repeated, the constituents of the singularity are* $C = 15$, $B = 8$.

Consider the surface

$$\alpha u_2^2 + u_5 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (53),$$

then, proceeding according to the first method, the two cones whose common generators determine the class are the sextic cones

$$u_2 u_5' = u_2' u_5,$$
$$u_2 u_5'' = u_2'' u_5,$$

which possess 36 common generators; but, since 10 of these common generators are the 10 lines of closest contact, which are the common generators of the cones u_2 and u_5, the total number of ordinary generators is 26. Hence

$$m = 26 = 80 - 2C - 3B,$$

whence

$$2C + 3B = 54 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (54).$$

Now, from § 9, it appears that a quadric cone twice repeated may be regarded as a quinquenodal quartic cone whose nodal generators lie on the quadric cone u_2, but are otherwise indeterminate; also these five nodal generators may be regarded as coinciding with five of the lines of closest contact. Hence

$$C + B = 18 + 5 = 23,$$

giving $C = 15$, $B = 8$.

The foregoing result is capable of extension to multiple points whose nodal cones are of the form $u_p^{\,r} u_q$.

Cuspidal Conics.

21. It is not my intention to discuss the properties of singular curves on surfaces, but I shall conclude by a revision and extension of *S. S.* §§ 49 and 50.

The equation of a quintic surface which has a nodal conic is

$$\alpha^2 (\alpha^3 U_0 + \alpha^2 U_1 + \alpha U_2 + U_3) + 2\alpha\Omega u_2 + \Omega^2 u = 0 \ldots (55),$$

in which $U_n = (\beta, \gamma, \delta)^n$; Ω and u_2 are quadric cones whose vertex is A, and u is the plane through A which cuts the plane α in the residual line. In the second of equations *S. S.* (46), Ω' ought to be written for Ω; and the forms of these equations show that the cones u_2, Ω', and U_3 pass through the points where the residual line cuts the cuspidal conic. Hence, if v and w be arbitrary planes through A, we may take

$$u_2 = uv + P\Omega,$$

$$\Omega' = uw + Q\Omega,$$

where P and Q are constants, accordingly

$$u_2^{\,2} - \Omega\Omega' = u^2 v^2 + 2P\Omega uv + P^2\Omega^2 - \Omega uw - Q\Omega^2 \ldots (56),$$

and, since the right-hand side of (56) has to be equal to $U_3 u$, it follows that $P^2 = Q$, and

$$U_3 = uv^2 + 2P\Omega v - \Omega w.$$

Substituting these values of u_2 and U_3 in (55) and writing $U_2' = U_2 + vw$, we obtain

$$\alpha^3 (\alpha^2 U_0 + \alpha U_1 + U_2') + (\alpha v + \Omega) \{(\alpha v + \Omega) u + 2F\Omega\alpha - \alpha^2 w\} = 0 \quad \ldots\ldots(57),$$

which shows that the quadric $\alpha v + \Omega = 0$ osculates the quintic along the cuspidal conic.

Since the points B, C, and D may be any points on the cuspidal conic, let them be three of the points of intersection of α, U_2', and Ω. Then it follows from (57) that the coefficients of β^3 and β^2 have a common factor, hence B is a tacnodal point; accordingly the cuspidal conic has four tacnodal points. We may now write (57) in the form

$$\alpha^3 (\alpha^2 U_0 + \alpha U_1 + U_2) + (\alpha v + \Omega) \{(\alpha v + \Omega) u + \alpha^2 w\} = 0 \ldots(58),$$

where v and w are arbitrary planes passing through A; and u is an arbitrary plane passing through the residual line. Also the tacnodal points are the intersections of the plane α with the cones U_2 and Ω.

22. If the quintic has a *tacnodal* conic, every point on the conic (α, Ω) must be a tacnodal point, which requires that $U_2 = Q\Omega$, and (58) may be reduced to the form

$$S^2 u + Sv\alpha^2 + w\alpha^4 = 0 \ldots\ldots\ldots\ldots(59),$$

where S is an arbitrary quadric and u, v, w arbitrary planes. This result will be obtained by a different method later on.

23. The form of (58) suggests that the general equation of a surface of the n^{th} degree, which has a cuspidal conic, is

$$\alpha^3 U_{n-3} + (\alpha v + \Omega) \{(\alpha v + \Omega) V_{n-4} + \alpha^2 W_{n-4}\} = 0 \ldots(60),$$

where v is an arbitrary plane and U_{n-3}, V_{n-4}, W_{n-4} are arbitrary quarternary quantics. This result may be verified as follows: let

$$v = p\alpha + q\beta + r\gamma + s\delta,$$

$$\Omega = \lambda\gamma\delta + \mu\delta\beta + \nu\beta\gamma,$$

$$U_{n-3} = \beta^{n-3} u_0 + \beta^{n-4} u_1 + \ldots\ u_{n-3},$$

$$V_{n-4} = \beta^{n-4} v_0 + \beta^{n-5} v_1 + \ldots\ v_{n-4},$$

$$W_{n-4} = \beta^{n-4} w_0 + \ldots\ w_{n-4},$$

where $u_n \equiv v_n \equiv w_n \equiv (\alpha, \gamma, \delta)^n$; then the highest power of β is β^{n-2}, and its coefficient is $(q\alpha + \mu\delta + \nu\gamma)^2$, which shows that (60) has a cuspidal conic.

Let B be one of the points of intersection of U_{n-3} with the cuspidal conic; then $u_0 = 0$, and B is a tacnodal point. Hence:—*A cuspidal conic on a surface of the n^{th} degree has $2n-6$ tacnodal points, which are the points of intersection of the surface U_{n-3} with the cuspidal conic.*

The curve $\alpha = 0$, $V_{n-4} = 0$ is the residual curve; let B be one of the points in which it cuts the cuspidal conic, then $v_0 = 0$, and the highest power of β is the $(n-3)^{\text{rd}}$ and its coefficient is

$$\alpha^3 u_0 + (q\alpha + \mu\delta + \nu\gamma)^2 v_1 + (q\alpha + \mu\delta + \nu\gamma)\,\alpha^2 w_0,$$

which shows that B is a cubic node on the cuspidal conic, at which there is a cuspidal cubic cone. The cuspidal tangent plane is

$$q\alpha + \mu\delta + \nu\gamma = 0,$$

and the line in which it cuts the plane α is the tangent at B to the cuspidal conic. *It therefore follows that a cuspidal conic on a surface of the n^{th} degree has $2n-8$ points of the kind which Cayley calls "close points."*

Tacnodal Conics.

24. The equation

$$S^2 U + SS' V + S'^2 W = 0 \quad \ldots\ldots\ldots\ldots\ldots (61),$$

where S, S' are quadric surfaces and U, V, W quarternary quantics of degree $n-4$, represents a surface of the n^{th} degree having a twisted nodal quartic, which is the curve of intersection of the quadrics S, S'. When the quadrics have a common generator, the nodal curve becomes a twisted cubic and a straight line; I shall not, however, stop to examine these special curves, but proceed to consider the case in which S' consists of two coincident planes. Putting $S' = \alpha^2$, (61) becomes

$$S^2 U + SV\alpha^2 + W\alpha^4 = 0 \quad \ldots\ldots\ldots\ldots\ldots (62),$$

which represents a surface having a tacnodal conic.

The plane α cuts the surface in the tacnodal conic twice repeated, and in a residual curve of degree $n-4$. Let B be

one of their points of intersection; then, if

$$\left.\begin{aligned} S &= \beta v_1 + v_2, \\ U &= \beta^{n-5}u_1 + \beta^{n-6}u_2 + \dots\ u_{n-4}, \\ V &= \beta^{n-4}v_0 + \beta^{n-5}v_1 + \dots\ v_{n-4}, \\ W &= \beta^{n-4}w_0 + \beta^{n-5}w_1 + \dots\ w_{n-4}, \end{aligned}\right\} \dots\dots (62A),$$

where the suffixed letters denote ternary quantics of (α, γ, δ), the coefficient of the highest power of β is

$$v_1 (v_1 u_1 + \alpha^2 v_0),$$

so that B is a cubic node, whose nodal cone consists of a quadric cone and a plane touching the cone. In other words the nodal cone is an *improper* cubic cone having a *tacnodal* generator. These are the singular points which correspond to what Cayley calls "close points" on a cuspidal conic; and it is remarkable that the singular generator on the nodal cone is of the same character as the singular curve on the surface. The fact that this is the case when the singular curve is a cuspidal or a tacnodal conic suggests far-reaching analogies. The number of these singular points is $2n - 8$.

25. We shall now show that:—

There are $4n - 16$ *points on the tacnodal conic, such that any plane section through one of them has a rhamphoid cusp at the point; and these points are the points of intersection of the tacnodal conic with the surface* $V^2 = 4UW$.

Let B be one of the points of intersection of the surface $V^2 = 4UW$ and the tacnodal conic; and let C and D be any arbitrary points on the latter. Then the values of V and W will be given by $(62A)$, but we must add the term $u_0\beta^{n-4}$ to that of U. But, since the surface $V^2 = 4UW$ has to pass through B, it follows that $v_0^2 = 4u_0 w_0$; hence, changing constants, the section of the surface by the plane δ is a curve of the form

$$\begin{aligned} &(\nu\beta\gamma + Q\gamma\alpha + P\alpha\beta)^2\ (\beta^{n-5}U_1 + \dots\dots U_{n-4}) \\ &+ \alpha^2\ (\nu\beta\gamma + Q\gamma\alpha + P\alpha\beta)\ (\beta^{n-5}V_1 + \dots\dots\ V_{n-4}) \\ &+ \alpha^4\ (\beta^{n-5}W_1 + \beta^{n-6}W_2 + \dots\dots\ W_{n-4}) \\ &+ \beta^{n-4}\{(\nu\beta\gamma + Q\gamma\alpha + P\alpha\beta)\ p + q\alpha^2\}^2 = 0 \ \dots\dots\dots\dots\dots (63), \end{aligned}$$

where all the suffixed letters are binary quantics of (α, γ).

Let us now transform (63) by means of the equation

$$\frac{\beta}{\alpha'\gamma'} = \frac{\alpha}{\alpha'\beta'} = \frac{\gamma}{\gamma'\beta'},$$

then, from the theory of birational transformation explained in *S.C.* §§ 1 and 7, it follows that, if (63) has a rhamphoid cusp at B, the transformed curve must have an ordinary cusp on AC. Now, omitting accents, *the transformed* curve is

$$\begin{aligned}&\beta\gamma^2(P\alpha+Q\beta+R\gamma)^2(\alpha^{n-5}\gamma^{n-5}U_1+\ldots\ldots\ \beta^{n-5}U_{n-4})\\&+\alpha\beta^2\gamma\ (P\alpha+Q\beta+R\gamma)\ (\alpha^{n-5}\gamma^{n-5}V_1+\ldots\ldots\ \beta^{n-5}V_{n-4})\\&+\alpha^2\beta^2\ (\alpha^{n-5}\gamma^{n-5}W_1+\ldots\ldots\ \beta^{n-5}W_{n-4})\\&+\alpha^{n-4}\gamma^{n-4}\{p\gamma\ (P\alpha+Q\beta+R\gamma)+q\alpha\beta\}^2=0\ \ldots\ldots\ldots\ldots(64),\end{aligned}$$

which shows that (64) has a cusp at the point where the line

$$P\alpha+Q\beta+R\gamma=0\ \ldots\ldots\ldots\ldots\ldots\ldots(65)$$

cuts AC.

Additional information on Quintic Surfaces having a tac-nodal conic will be found in a paper shortly to be published in the *Rendiconti del Circolo Matematico di Palermo* 1907.

THE EXPANSIONS OF PRODUCTS OF HYPERGEOMETRIC FUNCTIONS.

By G. N. Watson, Scholar of Trinity College, Cambridge.

§ 1. THE expansion in a power series of the square of the hypergeometric function $F(\alpha, \beta, \gamma.x)$, when α, β, γ are connected by the relation

$$\alpha+\beta+\tfrac{1}{2}=\gamma$$

has been investigated by Goursat* in the cases in which γ is not an integer.

The objects of this paper are as follows: first, to obtain expansions for the products of the complete elliptic integrals usually denoted by K, K', E, E'; it being well known that

$$K=\tfrac{1}{2}\pi F(\tfrac{1}{2}, \tfrac{1}{2}, 1.k^2),$$

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* *Ann. de l'Ecole Normale Supérieure*, Ser. 2, Tome XII.

and consequently does not come under the head of cases investigated by Goursat; second, to investigate the squares of some hypergeometric functions not discussed by Goursat; and, third, to evaluate some definite integrals involving products of the complete elliptic integrals; some results of this nature have been previously given by Dr. J. W. Nicholson.*

§ 2. It has been proved by Appell† that if the general solution of the equation

$$\frac{d^2y}{dx^2} = f_1(x)\frac{dy}{dx} + f_2(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

be $$y = c_1y_1 + c_2y_2,$$

then the equation of which the general solution is

$$z = Ay_1^2 + By_1y_2 + Cy_2^2$$

is

$$\frac{d^3z}{dx^3} - 3f_1(x)\frac{d^2z}{dx^2} + \left[2\{f_1(x)\}^2 - \frac{d}{dx}f_1(x) - 4f_2(x)\right]\frac{dz}{dx}$$
$$+ \left[4f_1(x)f_2(x) - 2\frac{d}{dx}f_2(x)\right]z = 0 \quad \ldots\ldots\ldots(2).$$

It is proved by Goursat‡ that if (1) be the differential equation of the function $F(\alpha, \beta, \gamma.x)$, then, when $\alpha+\beta+\frac{1}{2}=\gamma$, (2) can be written in the form

$$[\vartheta(\vartheta+\gamma-1)(\vartheta+2\gamma-2) - x(\vartheta+2\alpha)(\vartheta+2\beta)(\vartheta+\alpha+\beta)]z = 0 \quad \ldots\ldots(3),$$

ϑ denoting $x\dfrac{d}{dx}$.

We shall only consider hypergeometric functions where the relation $\alpha+\beta+\frac{1}{2}=\gamma$ holds, except in § 7.

§ 3. We proceed to apply this result to the equation for the quarter periods K, K', viz.

$$\frac{d}{dc}\left\{c(1-c)\frac{dy}{dc}\right\} - \tfrac{1}{4}y = 0,$$

where $c = k^2$.

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* *Phil. Mag.*, 1905.
† *Comptes Rendus*, Tome XCI.
‡ *Loc. cit.*

We first change the independent variable in this equation. Putting $c = \frac{1}{2}(1+u)$, it becomes

$$\frac{d}{du}\left\{(1-u^2)\frac{dy}{du}\right\} - \tfrac{1}{4}y = 0,$$

i.e.

$$\frac{d}{du}\left\{(u-u^3)\frac{dy}{udu}\right\} - \tfrac{1}{4}y = 0,$$

i.e.

$$u^2(1-u^2)\frac{d}{udu}\left\{\frac{dy}{udu}\right\} + (1-3u^2)\frac{dy}{udu} - \tfrac{1}{4}y = 0.$$

Writing $1-u^2 = x$, we have

$$x(1-x)\frac{d^2y}{dx^2} + (1-\tfrac{3}{2}x)\frac{dy}{dx} - \tfrac{1}{16}y = 0,$$

where $x = 4k^2k'^2$.

This is the equation for the hypergeometric function $F(\frac{1}{4}, \frac{1}{4}, 1.x)$. The transformation is a particular case of one of Riemann's transformations.*

Consequently by (3) the equation, the solution of which is

$$z = AK^2 + BKK' + CK'^2,$$

is

$$[\vartheta^3 - x(\vartheta+\tfrac{1}{2})^3]z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since in this case $\gamma = 1$, some of the solutions involve logarithms, and have not been hitherto investigated.

§ 4. We proceed to solve equation (4) by the method of Frobenius†; it is found that

$$z' = c_0 x^\alpha \left[1 + \sum_{n=1}^{\infty}\left\{\frac{(\alpha+\frac{1}{2})(\alpha+\frac{3}{2})\ldots(\alpha+n-\frac{1}{2})}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}\right\}^3 x^n\right] \ldots (5),$$

is a solution of the equation

$$[\vartheta^3 - x(\vartheta+\tfrac{1}{2})^3]z' = c_0\alpha^3 x^\alpha \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

c_0 being an arbitrary constant; and consequently solutions of

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* *Mathem. Werke*, p. 76.

† Forsyth, *Differential Equations*, pp. 235–249.

(4) are given by

$$z=(z')_{\alpha=0},\quad z=\left(\frac{\partial z'}{\partial\alpha}\right)_{\alpha=0},\quad z=\left(\frac{\partial^2 z'}{\partial\alpha^2}\right)_{\alpha=0}.$$

Now

$$\frac{\partial z'}{\partial\alpha}=z'\log x+c_0x^\alpha\left[\sum_{n=1}^{\infty}3\left\{\frac{(\alpha+\frac12)\ldots(\alpha+n-\frac12)}{(\alpha+1)\ldots(\alpha+n)}\right\}^3\right.$$
$$\left.\times\left\{\frac{1}{\alpha+\frac12}+\ldots+\frac{1}{\alpha+n-\frac12}-\left(\frac{1}{\alpha+1}+\ldots+\frac{1}{\alpha+n}\right)\right\}x^n\right],$$

and

$$\frac{\partial^2 z'}{\partial\alpha^2}=z'(\log x)^2+2c_0x^\alpha\log x\left[\sum_{n=1}^{\infty}3\left\{\frac{(\alpha+\frac12)\ldots(\alpha+n-\frac12)}{(\alpha+1)\ldots(\alpha+n)}\right\}^3\right.$$
$$\left.\times\left\{\frac{1}{\alpha+\frac12}+\ldots+\frac{1}{\alpha+n-\frac12}-\left(\frac{1}{\alpha+1}+\ldots+\frac{1}{\alpha+n}\right)\right\}x^n\right]$$
$$+c_0x^\alpha\left[\sum_{n=1}^{\infty}9\left\{\frac{(\alpha+\frac12)\ldots(\alpha+n-\frac12)}{(\alpha+1)\ldots(\alpha+n)}\right\}^3\right.$$
$$\left.\times\left\{\frac{1}{\alpha+\frac12}+\ldots+\frac{1}{\alpha+n-\frac12}-\left(\frac{1}{\alpha+1}+\ldots+\frac{1}{\alpha+n}\right)\right\}^2x^n\right]$$
$$-c_0x^\alpha\left[\sum_{n=1}^{\infty}3\left\{\frac{(\alpha+\frac12)\ldots(\alpha+n-\frac12)}{(\alpha+1)\ldots(\alpha+n)}\right\}^3\right.$$
$$\left.\times\left\{\frac{1}{(\alpha+\frac12)^2}+\ldots+\frac{1}{(\alpha+n-\frac12)^2}-\left(\frac{1}{(\alpha+1)^2}+\ldots+\frac{1}{(\alpha+n)^2}\right)\right\}x^n\right].$$

That is to say, putting

$$c_0=1,\quad \alpha=0,\quad x=(2kk')^2,$$

and writing for the sake of brevity

$$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}=a_n\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

$$\frac11+\frac12+\ldots+\frac1n=s_n\ \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\ldots+\frac{1}{n^2}=\sigma_n\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

we find that each of the three expressions z_1, z_2, z_3, where

$$z_1 = 1 + \sum_{n=1}^{\infty} \{a_n^3 (2kk')^{2n}\}$$

$$z_2 = 2z_1 \log(2kk') + 6 \sum_{n=1}^{\infty} \{a_n^3 (s_{2n} - s_n)(2kk')^{2n}\},$$

$$z_3 = 4z_1 \{\log(2kk')\}^2 + 24 \{\log(2kk')\} \sum_{n=1}^{\infty} \{a_n^3 (s_{2n} - s_n)(2kk')^{2n}\} + 36 \sum_{n=1}^{\infty} \{a_n^3 (s_{2n} - s_n)^2 (2kk')^{2n}\} - 12 \sum_{n=1}^{\infty} \{a_n^3 (\sigma_{2n} - \tfrac{1}{2}\sigma_n)(2kk')^{2n}\},$$

is of the form $AK^2 + BKK' + CK'^2$.

Now each of the series for z_1, z_2, z_3 is absolutely convergent so long as $|2kk'| < 1$, *i.e.* putting $k^2 = c$, so long as $|c(1-c)| < \frac{1}{4}$; that is to say, so long as, in the Argand diagram, c lies within the Lemniscate of Bernoulli with foci $c = 0$ and $c = 1$; except that z_2, z_3 have logarithmic singularities at $c = 0$ and $c = 1$.

It is necessary to consider separately the values of z_1, z_2, z_3 in the two loops of the Lemniscate, since the series for z_1, z_2, z_3 may represent different analytic functions of k in the two loops; that is to say, if real values of k, k' only are considered, it is necessary to consider separately the cases when $0 < c < \frac{1}{2}$ and $\frac{1}{2} < c < 1$; the double point of the Lemniscate being at the point $c = \frac{1}{2}$.

First, suppose $0 < c < \frac{1}{2}$; *i.e.* $0 < k < 1/\sqrt{2}$. When k is very small

$$K = \tfrac{1}{2}\pi + \text{terms in } k^2,\ k^4,\ \ldots,$$

$$K' = \log 4/k + \text{terms in } k^2 \log k,\ k^4 \log k,\ \ldots,$$

and in $k^2,\ k^4,\ \ldots$.

Therefore

$$K^2 = \tfrac{1}{4}\pi^2 + \text{terms which vanish with } k,$$

$$KK' = \tfrac{1}{2}\pi \log 4/k + \quad ,, \quad ,,$$

$$K'^2 = (\log 4/k)^2 + \quad ,, \quad ,,$$

Now $z_1 = 1 +$ terms which vanish with k,

$$z_2 = 2\log 2 + 2\log k + \text{terms which vanish with } k,$$

$$z_3 = 4\{(\log k)^2 + 2\log 2 \log k + (\log 2)^2\} + \text{terms which vanish with } k.$$

Therefore, when $0 < k < 1/\sqrt{2}$,

$$K^2 = (\tfrac{1}{4}\pi^2)\left[1 + \sum_{n=1}^{\infty}\{a_n{}^3 (2kk')^{2n}\}\right] \ldots\ldots\ldots\ldots (\mathrm{I}a),$$

$$KK' = \{\tfrac{3}{2}\pi \log 2\}\, z_1 - (\tfrac{1}{4}\pi)\, z_2 \ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{II}),$$

$$K'^2 = 9\,(\log 2)^2 z_1 - 3\,(\log 2)\, z_2 + \tfrac{1}{4} z_3 \ldots\ldots\ldots (\mathrm{III}a).$$

Next suppose that $1/\sqrt{2} < k < 1$. When k' is very small,

$$K' = \tfrac{1}{2}\pi + \text{terms in } k'^2,\ k'^4,\ \ldots,$$

$$K = \log 4/k' + \text{terms in } k'^2 \log k',\ \ldots,$$

$$\text{and in } k'^2,\ \ldots.$$

Whence it follows, since z_1, z_2, z_3 are unaltered by interchanging k and k', that when $1/\sqrt{2} < k < 1$,

$$K^2 = 9\,(\log 2)^2 z_1 - 3\,(\log 2)\, z_2 + \tfrac{1}{4} z_3 \ldots\ldots\ldots (\mathrm{I}b),$$

KK' has the same value as above$\ldots\ldots\ldots$(II),

$$K'^2 = (\tfrac{1}{4}\pi^2)\, z_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\mathrm{III}b).$$

§ 5. The products which contain E or E' may be determined from a differential equation of the third order in a somewhat similar way; but it is simpler to deduce them from the differential equations connecting K and E, with the help of Legendre's relation, in some cases.

These equations are to be found in any text book on Elliptic Functions; they are as follows:

$$\frac{dK}{dk} = \frac{E}{kk'^2} - \frac{K}{k} \ldots\ldots\ldots\ldots\ldots (10),$$

$$\frac{dK'}{dk} = -\frac{E'}{kk'^2} + \frac{kK'}{k'^2} \ldots\ldots\ldots\ldots\ldots (11),$$

$$\frac{dE}{dk} = \frac{E - K}{k} \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

$$\frac{dE'}{dk} = \frac{k}{k'^2}(K' - E') \ldots\ldots\ldots\ldots (13),$$

$$KE' + K'E - KK' = \tfrac{1}{2}\pi \ldots\ldots\ldots\ldots (14).$$

From (10) we have

$$EK = k'^2 K^2 + \tfrac{1}{2} kk'^2 \frac{dK^2}{dk} \ldots\ldots\ldots\ldots (15).$$

From (11)

$$E'K' = k^2K'^2 - \tfrac{1}{2}kk'^2\,\frac{dK'^2}{dk} \quad \ldots\ldots\ldots\ldots(16).$$

From (10) and (11)

$$kk'^2\,\frac{dKK'}{dk} = EK' - E'K - KK'(k'^2 - k^2).$$

Using Legendre's relation, we have

$$EK' = \tfrac{1}{4}\pi + \tfrac{1}{2}kk'^2\,\frac{dKK'}{dk} + k'^2KK' \quad \ldots\ldots\ldots(17),$$

$$E'K = \tfrac{1}{4}\pi - \tfrac{1}{2}kk'^2\,\frac{dKK'}{dk} + k^2\,KK' \quad \ldots\ldots\ldots(18).$$

From (10) and (12)

$$E^2 = kk'^2\,\frac{dEK}{dk} + k'^2K^2 \quad \ldots\ldots\ldots\ldots(19).$$

From (11) and (13)

$$E'^2 = -kk'^2\,\frac{dE'K'}{dk} + k^2K'^2 \quad \ldots\ldots\ldots\ldots(20).$$

From (11) and (12)

$$EE' = EK' - k'^2KK' - kk'^2\,\frac{dEK'}{dk} \quad \ldots\ldots\ldots(21).$$

From these relations and equations (I*a*) to (III*b*) we can deduce series for the different products; those which involve logarithms are hardly worth writing down; the others are: when $0 < k < 1/\sqrt{2}$,

$$EK = \tfrac{1}{4}\pi^2\Big[k'^2 + \sum_{n=1}^{\infty} a_n{}^3\,(2kk')^{2n}\{k'^2 + n\,(k'^2 - k^2)\}\Big] \ldots(\mathrm{IV});$$

when $1/\sqrt{2} < k < 1$,

$$E'K' = \tfrac{1}{4}\pi^2\Big[k^2 + \sum_{n=1}^{\infty} a_n{}^3\,(2kk')^{2n}\{k^2 + n\,(k^2 - k'^2)\}\Big] \ldots(\mathrm{V});$$

when $0 < k < 1/\sqrt{2}$,

$$E^2 = \tfrac{1}{4}\pi^2\Big[k'^2\,(k'^2 - k^2) + \sum_{n=1}^{\infty} a_n{}^3\{2n^2\,(k'^2 - k^2)^2 + 2nk'^2\,(k'^2 - 3k^2) + k'^2\,(k'^2 - k^2)\}\,(2kk')^{2n}\Big] \ldots(\mathrm{VI});$$

when $1/\sqrt{2} < k < 1$,

$$E'^2 = \tfrac{1}{4}\pi^2\left[k^2(k^2-k'^2) + \sum_{n=1}^{\infty} a_n{}^3\{2n^2(k'^2-k^2)^3 + 2nk^2(k^2-3k'^2) + k^2(k^2-k'^2)\}(2kk')^{2n}\right] \ldots \text{(VII)}.$$

§ 6. The solutions of (3) given by Goursat may be written

$$\zeta_1 = \left[1 + \sum_{n=1}^{\infty} \frac{(2\alpha)_n (2\beta)_n (\gamma - \frac{1}{2})_n}{(2\gamma-1)_n (\gamma)_n . n!} x^n\right] \quad \ldots\ldots(22),$$

$$\zeta_2 = x^{1-\gamma}\left[1 + \sum_{n=1}^{\infty} \frac{(2\alpha-\gamma+1)_n (2\beta-\gamma+1)_n (\frac{1}{2})_n}{(2-\gamma)_n (\gamma)_n . n!} x^n\right] \quad \ldots\ldots(23),$$

$$\zeta_3 = x^{2-2\gamma}\left[1 + \sum_{n=1}^{\infty} \frac{(2\alpha-2\gamma+2)_n (2\beta-2\gamma+2)_n (\frac{3}{2}-\gamma)_n}{(3-2\gamma)_n (2-\gamma)_n . n!} x^n\right] \ldots(24),$$

where $(\alpha)_n$ denotes $\alpha(\alpha+1)\cdots(\alpha+n-1)$.

Since two independent solutions of the hypergeometric equation (if γ is not an integer and $\alpha+\beta+\frac{1}{2}=\gamma$) are the expressions

$$F(\alpha,\ \beta,\ \gamma.x),$$

$$x^{1-\gamma}F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma.x),$$

we see at once that if $1-\gamma$, $2-2\gamma$ are not integers

$$\zeta_1 = \{F(\alpha,\ \beta,\ \gamma.x)\}^2,$$

$$\zeta_2 = \{F(\alpha,\ \beta,\ \gamma.x)\}\, x^{1-\gamma}\{F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma.x)\},$$

$$\zeta_3 = x^{2-2\gamma}\{F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma.x)\}^2.$$

We now consider the case (which was not noticed by Goursat) when 2γ is an integer, but γ is not an integer.

First, suppose that $\gamma = \frac{3}{2} + m$ where m is zero or a positive integer.

Then ζ_3 becomes a polynomial going from x^{-2m-1} to x^{-m-1}, consequently

$$\zeta_1 = \{F(\alpha,\ \beta,\ \gamma.x)\}^2,$$

$$\zeta_2 = x^{1-\gamma}\{F(\alpha,\ \beta,\ \gamma.x)\}\{F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma.x),$$

as before, but we must have a relation of the form

$$\zeta_3 + C_1\zeta_1 = x^{-2m-1}\{F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma.x)\}^2,$$

i.e.

$$\zeta_3 + C_1\{F(\alpha, \beta, \gamma.x)\}^2 = x^{-2m-1}\{F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}-\beta, 2-\gamma.x)\}^2 \quad \ldots\ldots(25),$$

where C_1 is a constant to be determined.

Now

$$F(\alpha, \beta, \gamma.x) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(1-x)^{-\alpha}F\left(\alpha, \gamma-\beta, \alpha-\beta+1.\frac{1}{1-x}\right)$$

$$+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(1-x)^{-\beta}F\left(\beta, \gamma-\alpha, \beta-\alpha+1.\frac{1}{1-x}\right),$$

$$F(\tfrac{1}{2}-\alpha, \tfrac{1}{2}-\beta, 2-\gamma.x)$$

$$=\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\frac{1}{2}-\beta)\Gamma(\frac{3}{2}-\gamma+\alpha)}(1-x)^{\alpha-\frac{1}{2}}F\left(\tfrac{1}{2}-\alpha, 1-\alpha, \beta-\alpha+1.\frac{1}{1-x}\right)$$

$$+\frac{\Gamma(2-\gamma)\Gamma(\beta-\alpha)}{\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{3}{2}-\gamma+\beta)}(1-x)^{\beta-\frac{1}{2}}F\left(\tfrac{1}{2}-\beta, 1-\beta, \alpha-\beta+1.\frac{1}{1-x}\right),$$

(Forsyth's *Differential Equations*).

These results are valid, if α, β are real and unequal and are not integers or halves of odd integers, in the common region of convergence of the series, provided we make $(1-x)^{-\alpha}$, $(1-x)^{-\beta}$, $(1-x)^{\alpha-\frac{1}{2}}$, $(1-x)^{\beta-\frac{1}{2}}$ tend to unity as x tends to zero.

Suppose $\alpha > \beta$, and substitute these results in (25). Then we find that on the part of the real axis between 0 and -1 an expression involving the products of some hypergeometric functions of argument $(1-x)^{-1}$ is equal to a polynomial ζ_3.

Now make x tend to ∞ along the negative part of the real axis; the hypergeometric functions are regular all along the path of x, and the path of x does not go round any branch points of any of the functions involved.

Then ζ_3 tends to zero, and we see, by considering the highest powers of x, that

$$C_1\left\{\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}\right\}^2(1-x)^{-2\beta}$$

tends to equality with

$$x^{-2m-1}(1-x)^{2\alpha-1}\left\{\frac{\Gamma(2-\gamma)\Gamma(\alpha-\beta)}{\Gamma(\frac{1}{2}-\beta)\Gamma(\frac{3}{2}-\gamma+\alpha)}\right\}^2.$$

Consequently

$$C_1 = -\left\{\frac{\Gamma(2-\gamma)\,\Gamma(\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\frac{1}{2}-\beta)\,\Gamma(1-\beta)\,\Gamma(\gamma)}\right\}^2.$$

And therefore, when $\alpha > \beta$,

$$\{F(\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma.x)\}^2 = 1+\sum_{n=1}^{\gamma-\frac{3}{2}} \frac{(2\alpha-2\gamma+2)_n (2\beta-2\gamma+2)_n (\frac{3}{2}-\gamma)_n}{(3-2\gamma)_n (2-\gamma)_n n!} x^n$$

$$-\left\{\frac{\Gamma(2-\gamma)\,\Gamma(\alpha)\,\Gamma(\gamma-\beta)}{\Gamma(\frac{1}{2}-\beta)\,\Gamma(1-\beta)\,\Gamma(\gamma)}\right\}^2 \left[1+\sum_{n=1}^{\infty} \frac{(2\alpha)_n (2\beta)_n (\gamma-\frac{1}{2})_n}{(2\gamma-1)_n (\gamma)_n n!} x^n\right] x^{2\gamma-2}$$

.....(26).

Similarly, if $\gamma = \frac{1}{2} - m$, m being a positive integer or zero, writing $\frac{1}{2}-\beta$, $\frac{1}{2}-\alpha$, $2-\gamma$ for α, β, γ in the last result we see that, when $\alpha > \beta$,

$$\{F(\alpha,\ \beta,\ \gamma.x)\}^2 = 1+\sum_{n=0}^{\frac{1}{2}-\gamma} \frac{(2\alpha)_n (2\beta)_n (\gamma-\frac{1}{2})_n}{(2\gamma-1)_n (\gamma)_n n!} x^n$$

$$-\frac{\Gamma(\gamma)\Gamma(\frac{1}{2}-\beta)\Gamma(1-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)\Gamma(2-\gamma)}\left[1+\sum_{n=1}^{\infty} \frac{(2\alpha-2\gamma+2)_n (2\beta-2\gamma+2)_n (\frac{3}{2}-\gamma)_n}{(3-2\gamma)_n (2-\gamma)_n n!} x^n\right] x^{2-2\gamma}$$

.....(27).

If $\gamma = \frac{1}{2}$,

$$F\{\alpha,\ \beta,\ \gamma.\sin^2\theta\} = \cos 2\alpha\theta,$$

$$F\{\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta,\ 2-\gamma,\ \sin^2\theta\} = \frac{\sin 2\alpha\theta}{2\alpha\sin\theta},$$

and we merely obtain some elementary trigonometrical formulæ.

§ 7. We proceed to determine the other cases in which $\{F(\alpha,\ \beta,\ \gamma.x)\}^2$ can be expanded in any sort of power series, when $\alpha+\beta+\frac{1}{2}-\gamma \neq 0$.

For convenience, we shall write the power series given by ζ_1, ζ_2, ζ_3 in the forms

$$\Phi_1(\alpha,\ \beta,\ \gamma.x);\ x^{1-\gamma}\Phi_2(\alpha,\ \beta,\ \gamma.x);\ x^{2-2\gamma}\Phi_3(\alpha,\ \beta,\ \gamma.x).$$

Case I. $\alpha+\beta-\frac{1}{2}=\gamma$.

Then

$$F(\alpha, \beta, \gamma . x) = (1-x)^{\gamma-\alpha-\beta} F(\gamma-\alpha, \gamma-\beta, \gamma . x),$$

$$x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)$$
$$= x^{1-\gamma}(1-x)^{\gamma-\alpha-\beta} F(1-\alpha, 1-\beta, 2-\gamma . x);$$

consequently

$$\{F(\alpha, \beta, \gamma . x)\}^2 = (1-x)^{-1}\{F(\gamma-\alpha, \gamma-\beta, \gamma . x)\}^2$$
$$= (1-x)^{-1} \Phi_1(\gamma-\alpha, \gamma-\beta, \gamma . x) \ldots (28).$$

Similarly

$$F(\alpha, \beta, \gamma . x) F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)$$
$$= (1-x)^{-1} \Phi_2(\gamma-\alpha, \gamma-\beta, \gamma . x) \ldots (29),$$

$$\{F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)\}^2$$
$$= (1-x)^{-1} \Phi_3(\gamma-\alpha, \gamma-\beta, \gamma . x) \ldots (30).$$

Case II. $\alpha-\beta+\frac{1}{2}=0.$

Then

$$F(\alpha, \beta, \gamma . x) = (1-x)^{-\alpha} F\left(\alpha, \gamma-\beta, \gamma . \frac{x}{x-1}\right),$$

$$x^{1-\gamma} F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)$$
$$= x^{1-\gamma}(1-x)^{\gamma-\alpha-1} F\left(\alpha-\gamma+1, 1-\beta, 2-\gamma . \frac{x}{x-1}\right),$$

and in a similar manner to case I.

$$\{F(\alpha, \beta, \gamma . x)\}^2 = (1-x)^{-2\alpha} \Phi_1\left(\alpha, \gamma-\beta, \gamma . \frac{x}{x-1}\right) \ldots (31),$$

$$\{F(\alpha, \beta, \gamma . x)\}\{F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)\}$$
$$= (1-x)^{\gamma-2\alpha-1} \Phi_2\left(\alpha, \gamma-\beta, \gamma . \frac{x}{x-1}\right) \ldots (32).$$

$$\{F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)\}^2$$
$$= (1-x)^{2\gamma-2\alpha-2} \Phi_3\left(\alpha, \gamma-\beta, \gamma . \frac{x}{x-1}\right);$$

this equation is practically the same as (31).

These results hold only in the common part of the two regions determined by $|x|<1$, $|x|<|x-1|$.

Case III. $\beta - \alpha + \frac{1}{2} = 0$.

This is obtained from case II. by interchanging α and β.

There are several other cases of a similar, but more complicated, nature.

Following the notation of Forsyth's *Differential Equations*, Chapter VI., we have, if $y_1, y_2, \ldots, y_{23}, y_{24}$ be the twenty-four solutions of the hypergeometric equation

$$y_1 = y_2 = y_{17} = y_{18} = Y_1,$$
$$y_3 = y_4 = y_{19} = y_{20} = Y_2,$$
$$y_5 = y_6 = y_{21} = y_{22} = Y_3,$$
$$y_7 = y_8 = y_{23} = y_{24} = Y_4,$$
$$y_9 = y_{12} = y_{13} = y_{15} = Y_5,$$
$$y_{10} = y_{11} = y_{14} = y_{16} = Y_6.$$

There is a linear relation between any three of the Y's, but not between any pair.

We consequently have relations such as

$$y_1 = ay_5 + by_7,$$
$$y_3 = a'y_5 + b'y_7;$$

therefore, if y_5^2, y_5y_7, y_7^2 can be expanded in power series of their argument, $1-x$, then y_1^2, y_1y_3, y_3^2 can also be expanded in power series of argument $1-x$.

A table of linear relations between α, β, γ and the corresponding arguments in which y_1^2, y_1y_3, y_3^2 can be expanded can be written down.

Relation between α, β, γ.	Argument of power series.	
$\gamma - \frac{1}{2} = 0$	$1-x$	y_5, y_7
$\gamma - \frac{3}{2} = 0$	$1-x$	y_6, y_8
$\gamma - \frac{1}{2} = 0$	$(1-x)^{-1}$	y_{13}, y_{14}
$\gamma - \frac{3}{2} = 0$	$(1-x)^{-1}$	y_{15}, y_{16}
$\alpha - \beta + \frac{1}{2} = 0$	$\frac{x-1}{x}$	y_{21}, y_{24}
$\beta - \alpha + \frac{1}{2} = 0$	$\frac{x-1}{x}$	y_{22}, y_{23}

The expressions in the third column being those in terms of which $F(\alpha, \beta, \gamma . x)$ and $x^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma . x)$ are expressed.

§ 8. We now proceed to obtain a power series for $\{P_n(\mu)\}^2$.

So far as I know, the only series hitherto given for $\{P_n(\mu)\}^2$ are (1) a series of Legendre functions* and (2) a series in powers of ξ^2,† where $2\mu = \xi + \xi^{-1}$.

The equation for $P_n(\mu)$ is

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dy}{d\mu}\right\} + n(n+1)\,y = 0,$$

i.e.
$$\frac{d}{d\mu}\left\{(\mu-\mu^3)\frac{dy}{\mu d\mu}\right\} + n(n+1)\,y = 0,$$

i.e.
$$\mu^2(1-\mu^2)\frac{d}{\mu d\mu}\left(\frac{dy}{\mu d\mu}\right) + (1-3\mu^2)\frac{dy}{\mu d\mu} + n(n+1)\,y = 0.$$

Writing $1-\mu^2 = x$, this becomes

$$x(1-x)\frac{d^2y}{dx^2} + (1-\tfrac{3}{2}x)\frac{dy}{dx} + \tfrac{1}{4}n(n+1)\,y = 0.$$

A solution of this equation is

$$y = AF\{-\tfrac{1}{2}n,\ \tfrac{1}{2}(n+1),\ 1\,.\,1-\mu^2\},$$

and the other solution has a singularity where $\mu^2 = 1$.

Consequently

$$P_n(\mu) = AF\{-\tfrac{1}{2}n,\ \tfrac{1}{2}(n+1),\ 1\,.\,1-\mu^2\},$$

when $|1-\mu^2| < 1$.

The hypergeometric series, however, has a branch point at $\mu^2 = 0$, and consequently (since in the μ plane the hypergeometric series converges within a Lemniscate, foci ± 1), when the real part of μ is positive

$$P_n(\mu) = A_1F\{-\tfrac{1}{2}n,\ \tfrac{1}{2}(n+1),\ 1\,.\,1-\mu^2\},$$

when the real part of μ is negative

$$P_n(\mu) = A_2F\{-\tfrac{1}{2}n,\ \tfrac{1}{2}(n+1),\ 1\,.\,1-\mu^2\}.$$

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* Adams, *Proc. Roy. Soc.*, 1878.

† Heine, *Kugelfunctionen*, Band II., p. 358.

Putting $\mu = 1$, $\mu = -1$ respectively, we get

$$1 = A_1,$$

$$(-1)^n = A_2,$$

n being a positive integer.

Since $-\frac{1}{2}n + \frac{1}{2}(n+1) + \frac{1}{2} = 1$, we have

$$\{P_n(\mu)\}^2 = \Phi_1\{-\tfrac{1}{2}n,\ \tfrac{1}{2}(n+1),\ 1\,.\,1-\mu^2\}$$

$$= 1 + \sum_{r=1}^{n} \frac{(-n)_r\,(n+1)_r\,(\frac{1}{2})_r}{(r\,!)^3}(1-\mu^2)^r \ldots\ldots(33).$$

In a similar manner we find for the generalised Legendre function $P_n^m(\mu)$

$$P_n^m(\mu) = A\,(1-\mu^2)^{\frac{1}{2}m}\,F\{\tfrac{1}{2}(m-n),\ \tfrac{1}{2}(m+n+1),\ m+1\,.\,1-\mu^2\}.$$

Now $P_n^m(\mu) = (1-\mu^2)^{\frac{1}{2}m} \times$ the coefficient of r^{n-m} in the expansion of

$$\frac{1.3.5\ldots(2m-1)}{(1-2r\mu+r^2)^{m+\frac{1}{2}}}.$$

Hence, by making μ tend to unity, we see that when the real part of μ is positive

$$A = \frac{(n+m)\,!}{2^m.m\,!\,(n-m)\,!},$$

and when the real part of μ is negative

$$A = (-)^{n-m}\,\frac{(n+m)\,!}{2^m.m\,!\,(n-m)\,!}.$$

Therefore, since

$$\tfrac{1}{2}(m-n) + \tfrac{1}{2}(m+n+1) + \tfrac{1}{2} = m+1,$$

$$\{P_n^m(\mu)\}^2 = \left\{\frac{(n+m)\,!}{2^m.m\,!\,(n-m)\,!}\right\}^2 (1-\mu^2)^m$$

$$\times\,\Phi_1\{\tfrac{1}{2}(m-n),\ \tfrac{1}{2}(m+n+1),\ m+1,\ 1-\mu^2\},$$

i.e.

$$\{P_n^m(\mu)\}^2 = \left\{\frac{(n+m)\,!}{2^m.m\,!\,(n-m)\,!}\right\}^2 \left[(1-\mu^2)^m\right.$$

$$\left. + \sum_{r=1}^{n-m} \frac{(m-n)_r\,(m+n+1)_r\,(m+\frac{1}{2})_r}{r\,!\,(m+1)_r\,(2m+1)_r}(1-\mu^2)^{m+r}\right] \ldots(34).$$

§ 9. We now proceed to evaluate some definite integrals connected with K, E.

a_n as defined in (7) $= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \sin^{2n}\theta d\theta.$

Consequently by (Ia), when $0 < k < 1/\sqrt{2}$,

$$\frac{4}{\pi^2}K^2 = \frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\left[1 + \sum_{n=1}^{\infty} a_n \sin^{2n}\theta \sin^{2n}\phi . (2kk')^{2n}\right] d\theta d\phi,$$

i.e. $$K^2 = \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta d\phi}{\left[1 - 4k^2k'^2\sin^2\theta\sin^2\phi\right]^{\frac{1}{2}}}.$$

Writing $$K(\lambda) = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{(1 - \lambda^2\sin^2\theta)^{\frac{1}{2}}},$$

we may write this in the form

$$\{K(k)\}^2 = \int_0^{\frac{1}{2}\pi} K(2kk'\sin\phi)\, d\phi \ldots\ldots\ldots\ldots (35),$$

when $0 < k < 1/\sqrt{2}$.

Also from (Ia) we have

$$\frac{4}{\pi^2}\int_0^{1/\sqrt{2}} K^2 k dk = \int_0^{1/\sqrt{2}} \left\{1 + \sum_{n=1}^{\infty} a_n{}^3 (2kk')^{2n}\right\} k dk$$

$$= \int_0^{\frac{1}{4}\pi} \left\{1 + \sum_{n=1}^{\infty} a_n{}^3 \sin^{2n} 2\theta\right\} \sin\theta\cos\theta d\theta$$

$$= \tfrac{1}{4}\int_0^{\frac{1}{2}\pi} \left(1 + \sum_{n=1}^{\infty} a_n{}^3 \sin^{2n}\phi\right) \sin\phi d\phi$$

$$= \tfrac{1}{4}\left[1 + \sum_{n=1}^{\infty} \frac{a_n{}^2}{2n+1}\right]$$

$$= \tfrac{1}{4}\int_0^1 \left[1 + \sum_{n=1}^{\infty} a_n{}^2 k^{2n}\right] dk$$

$$= \frac{1}{2\pi}\int_0^1 K dk.$$

By a known result, given by Torelli,*

$$\int_0^1 K dk = 2\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right).$$

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* *Giornale di Mathematiche*, Vol. x., 1872.

Consequently

$$\int_0^{1/\sqrt{2}} kK^2dk = \tfrac{1}{4}\pi\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\right)\ldots\ldots\ldots(36).$$

Also

$$\int_0^{1/\sqrt{2}} kK'^2dk = \int_{1/\sqrt{2}}^1 k'K'^2dk'$$

$$= \int_{1/\sqrt{2}}^1 kK^2dk.$$

Now it has been proved* that

$$\int_0^1 kK^2dk = 2\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\ldots\right),$$

and therefore

$$\int_0^{1/\sqrt{2}} kK'^2dk = 2\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\ldots\right)$$

$$-\tfrac{1}{4}\pi\left(1-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\right)\ldots\ldots\ldots(37).$$

While

$$\int_0^{1/\sqrt{2}} kKK'dk = \int_{1/\sqrt{2}}^1 k'KK'dk'$$

$$= \int_{1/\sqrt{2}}^1 kK'Kdk$$

$$= \tfrac{1}{2}\int_0^1 kKK'dk.$$

And therefore, since*

$$\int_0^1 kKK'dk = \tfrac{1}{16}\pi^3,$$

it follows that

$$\int_0^{1/\sqrt{2}} kKK'dk = \tfrac{1}{32}\pi^3\ldots\ldots\ldots\ldots\ldots(38).$$

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* Dr. Nicholson, *loc. cit.*

Before attempting to evaluate integrals involving the other products, it will be found convenient to determine a reduction formula for the integral

$$I_n = 8\int_0^{1/\sqrt{2}} (2kk')^{2n} K^2 k\,(k'^2 - k^2)\,dk.$$

Putting $K^2 = y$, $(2kk')^2 = x$, denoting the corresponding indefinite integral by I_n', and writing $\frac{dy}{dx} = \dot{y}$, etc., we have, from (4),

$$I_n = \int x^n [8x^2(1-x)\,\ddot{y} + (24 - 36x)\,x\ddot{y} + (8 - 26x)\,\dot{y}]\,dx.$$

By repeated integration by parts, we have, if $n > 0$,

$$\begin{aligned} I_n' = {} & 8x^{n+2}(1-x)\,\ddot{y} - \dot{y}\,\frac{d}{dx}\{8x^{n+2}(1-x)\} + y\,\frac{d^2}{dx^2}\{8x^{n+2}(1-x)\} \\ & + x^{n+1}(24 - 36x)\,\dot{y} - y\,\frac{d}{dx}\{x^{n+1}(24 - 36x)\} \\ & + y\,(8 - 26x)\,x^n \\ & - 8n^3 I'_{n-1} + (8n^3 + 12n^2 + 6n + 2)\,I_n. \end{aligned}$$

That is to say,

$$\begin{aligned} (2n)^3 I'_{n-1} - (2n+1)^3 I_n' = {} & x^n[8x^2(1-x)\,\ddot{y} - 8nx(1-x)\,\dot{y} \\ & + (8x - 12x^2)\,\dot{y} + 8n^2 y - (8n^2 + 4n + 2)\,xy]. \end{aligned}$$

Now

$$\frac{dy}{dx} = \frac{1}{8k\,(k'^2 - k^2)}\,\frac{dy}{dk},$$

$$\frac{d^2y}{dx^2} = \frac{1}{64k^3(k'^2 - k^2)^2}\,\frac{d^2y}{dk^2} - \frac{1 - 6k^2}{64k^3(k'^2 - k^2)^3}\,\frac{dy}{dk}.$$

And therefore

$$\begin{aligned} & (2n)^3 I'_{n-1} - (2n+1)^3 I_n' \\ & = (2kk')^{2n}[2k^2k'^4\,\frac{d^2y}{dk^2} + 2kk'^2(1 - 3k^2)\,\frac{dy}{dk} - 4nkk'^2(k'^2 - k^2)\,\frac{dy}{dk} \\ & \qquad + 8n^2 y - 4\,(8n^2 + 4n + 2)\,k^2k'^2 y]\ldots\ldots(39). \end{aligned}$$

Also
$$\frac{d^2y}{dk^2} = 2K\frac{d^2K}{dk^2} + 2\left(\frac{dK}{dk}\right)^2$$
$$= 2\left(\frac{k'^2-k^2}{k^2k'^2}\right)K^2 - 2\left(\frac{1-3k^2}{k^2k'^4}\right)EK$$
$$+ \frac{2}{k^2k'^4}E^2 - \frac{4}{k^2k'^2}EK + \frac{2}{k^2}K^2,$$
$$\frac{dy}{dk} = \frac{2}{kk'^2}EK - \frac{2}{k}K^2.$$

Therefore
$$(2n)^3 I'_{n-1} - (2n+1)^3 I'_n$$
$$= (2kk')^{2n}\left[4E^2 - 8k'^2EK - 8n(k'^2-k^2)EK\right.$$
$$\left. + \{4k'^2 + 8nk'^2(k'^2-k^2) + 8n^2 - 4k^2k'^2(8n^2+4n+2)\}K^2\right] \ldots (40).$$

A result which may be easily verified by differentiation; while, if $n=0$,
$$-I'_0 = 4E^2 - 8k'^2EK + (4k'^2 - 8k^2k'^2)K^2 \ldots\ldots (41).$$

Now equations (40) and (41) hold for all values of k between 0 and 1.

If, therefore, we integrate from 0 to 1, since k'^2K^2 tends to zero as k tends to unity,
$$8(2n)^3\int_0^1 (2kk')^{2n-2}K^2k(k'^2-k^2)\,dk$$
$$= 8(2n+1)^3\int_0^1 (2kk')^{2n}K^2k(k'^2-k^2)\,dk, \text{ if } n \geqslant 1;$$

while, if $n=0$,
$$-8\int_0^1 K^2k(k'^2-k^2)\,dk = 4\left[\{E(1)\}^2 - \{E(0)\}^2\right] + 8E(0)K(0) - 4\{K(0)\}^2$$
$$= 4.$$

Therefore
$$\int_0^1 (2kk')^{2n}K^2k(k^2-k'^2)\,dk = \tfrac{1}{2}\left\{\frac{2.4\ldots 2n}{3.5\ldots(2n+1)}\right\}^3 \ldots (42).$$

If, however, we integrate from 0 to $1/\sqrt{2}$, we get
$$(2n)^3 I_{n-1} - (2n+1)^3 I_n$$
$$= 4\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2 - 4E\left(\frac{1}{\sqrt{2}}\right)K\left(\frac{1}{\sqrt{2}}\right) - 4n\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2,$$

a reduction formula; while, if $n = 0$,

$$-I_0 = 4\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2 - 4E\left(\frac{1}{\sqrt{2}}\right)K\left(\frac{1}{\sqrt{2}}\right) \ldots\ldots (43).$$

We now consider

$$\int_0^{1/\sqrt{2}} EKkdk = \int_0^{1/\sqrt{2}} \left\{k'^2K^2 + kk'^2K\frac{dK}{dk}\right\} kdk, \text{ by (10)},$$

$$= \left[\tfrac{1}{2}k^2k'^2K^2\right]_0^{1/\sqrt{2}} + \int_0^{1/\sqrt{2}} (kk'^2 - k + 2k^3)K^2kdk$$

$$= \tfrac{1}{8}\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2 + \int_0^{1/\sqrt{2}} k^3K^2dk$$

$$= \tfrac{1}{8}\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2 + \tfrac{1}{2}\int_0^{1/\sqrt{2}} kK^2dk - \tfrac{1}{2}\int_0^{1/\sqrt{2}} k(k'^2 - k^2)K^2dk,$$

and therefore by (36) and (43)

$$\int_0^{1/\sqrt{2}} EKkdk = \tfrac{1}{8}\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2 - \tfrac{1}{4}E\left(\frac{1}{\sqrt{2}}\right)K\left(\frac{1}{\sqrt{2}}\right) + \tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2$$

$$+ \tfrac{1}{8}\pi\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right) \ldots\ldots\ldots\ldots\ldots (44),$$

$$= \tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2 - 1\tfrac{1}{6}\pi + \tfrac{1}{8}\pi\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right\} \text{ by (14)}.$$

Further

$$\frac{d}{dk}(k^2E^2) = 2kE^2 + 2k^2E\frac{E-K}{k} = 4kE^2 - 2kEK.$$

Integrating, we have

$$\int_0^{1/\sqrt{2}} E^2kdk = \tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2 - \tfrac{1}{32}\pi$$

$$+ 1\tfrac{1}{6}\pi\left(\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \ldots\right) \ldots\ldots (45).$$

Using another result of Dr. Nicholson's, viz.

$$\int_0^1 E^2kdk = \tfrac{1}{32}\left(\frac{1}{1^3} + \frac{1}{3^3} + \ldots\right),$$

we have

$$\int_0^{1/\sqrt{2}} E'^2 k\,dk = \tfrac{1}{32}\left(\frac{1}{1^3}+\frac{1}{3^3}+\ldots\right)-\tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2+\tfrac{1}{32}\pi$$
$$-\tfrac{1}{16}\pi\left(\frac{1}{1^2}-\frac{1}{3^2}+\ldots\right)\ldots\ldots\ldots\ldots\ldots(46).$$

And in a similar manner

$$\int_0^{1/\sqrt{2}} E'K'\,k\,dk = \tfrac{1}{16}+\tfrac{1}{4}\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\ldots\right)-\tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2$$
$$-\tfrac{1}{8}\pi\left(\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\ldots\right)+\tfrac{1}{16}\pi\ldots\ldots(47).$$

Some of these results can be put into a rather different form by making use of Legendre's relation

$$2E\left(\frac{1}{\sqrt{2}}\right)K\left(\frac{1}{\sqrt{2}}\right)-\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2=\tfrac{1}{2}\pi,$$

together with the result

$$K\left(\frac{1}{\sqrt{2}}\right)=\frac{\{\Gamma(\frac{1}{4})\}^2}{4\sqrt{\pi}},$$

also given by Legendre.

Now (39), when written in the form

$$8(2n)^3\int(2kk')^{2n-2}yk(k'^2-k^2)\,dk-8(2n+1)^3\int(2kk')^{2n}yk(k'^2-k^2)\,dk$$
$$=(2kk')^{2n}\left[2k^2k'^4\frac{d^2y}{dk^2}+2kk'^2(1-3k^2)\frac{dy}{dk}-4nkk'^2(k'^2-k^2)\frac{dy}{dk}\right.$$
$$\left.+8n^2y-4(8n^2+4n+2)k^2k'^2y\right],$$

holds if y be any solution of (4).

Put $y=KK'$, then since

$$\frac{dy}{dk}=\frac{EK'-KE'+(k^2-k'^2)KK'}{kk'^2},$$

$$\frac{d^2y}{dk^2}=\frac{1}{k^2k'^4}[-2EE'+EK'(5k^2-1)+E'K(3-5k^2)$$
$$+KK'(1-2k^2)(1-3k^2)],$$

we have

$$8(2n)^3\int(2kk')^{2n-2}yk(k'^2-k^2)dk-8(2n+1)^3\int(2kk')^{2n}yk(k'^2-k^2)dk$$
$$=(2kk')^{2n}[-4EE'+4k^2EK'+4k'^2E'K-4n(k'^2-k^2)(EK'-KE')$$
$$+\{8n^2+4n(k'^2-k^2)^2-4(8n^2+4n+2)k^2k'^2KK'].$$

Taking this between the limits 0 and $1/\sqrt{2}$, we get

$$8(2n)^3\int_0^{1/\sqrt{2}}(2kk')^{2n-2}KK'k(k'^2-k^2)\,dk$$
$$-8(2n+1)^3\int_0^{1/\sqrt{2}}(2kk')^{2n}KK'k(k'^2-k^2)\,dk$$
$$=-4\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2+4E\left(\frac{1}{\sqrt{2}}\right)K\left(\frac{1}{\sqrt{2}}\right)-(4n+2)\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2.$$

And

$$\int_0^{1/\sqrt{2}}KK'k(k'^2-k^2)\,dk=\tfrac{1}{2}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2-\tfrac{1}{8}\pi\ldots.(48).$$

Using the same procedure as in proving (38), we have

$$\int_0^{1/\sqrt{2}}EE'kdk=\tfrac{1}{2}\int_0^1EE'kdk=\frac{\pi^3}{2048}\ldots\ldots.(49)$$

by another of Dr. Nicholson's results.

Also

$$\frac{d}{dk}\{k^2k'^2KK'\}=-k(k'^2-k^2)\,KK'+(EK'-E'K)\,k.$$

Integrating, we get

$$\tfrac{1}{4}\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2=-\tfrac{1}{2}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2+\tfrac{1}{8}\pi$$
$$+\int_0^{1/\sqrt{2}}(EK'-E'K)\,kdk;$$

while, if we integrate Legendre's relation, we have

$$\int_0^{1/\sqrt{2}}(EK'+E'K)\,kdk=\tfrac{1}{8}\pi+\tfrac{1}{32}\pi^3.$$

Therefore

$$\int_0^{1/\sqrt{2}} EK'k\,dk = \tfrac{1}{64}\pi^3 + \tfrac{1}{8}\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2 + \tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2 \quad \ldots(50),$$

$$\int_0^{1/\sqrt{2}} E'Kk\,dk = \tfrac{1}{8}\pi + \tfrac{1}{64}\pi^3 - \tfrac{1}{8}\left\{K\left(\frac{1}{\sqrt{2}}\right)\right\}^2 - \tfrac{1}{4}\left\{E\left(\frac{1}{\sqrt{2}}\right)\right\}^2$$

$\ldots\ldots\ldots(51).$

§ 10. We shall now consider some more complicated integrals.

If we differentiate (40) with regard to n and then put $n=0$, we have

$$-6I_0' - \left(\frac{\partial}{\partial n} I_n'\right)_{n=0} = 2\log(2kk')\left[4E^2 - 8k'^2EK + (4k'^2 - 8k^2k'^2)K^2\right]$$
$$+ \left[-8(k'^2-k^2)EK + \{8k'^2(k'^2-k^2) - 16k^2k'^2\}K^2\right].$$

That is to say

$$-48\int K^2k(k'^2-k^2)\,dk - 8\int 2(\log 2kk')K^2k(k'^2-k^2)\,dk$$
$$= 2\log(2kk')\left[4E^2 - 8k'^2EK + (4k'^2 - 8k^2k'^2)K^2\right]$$
$$+ \left[-8(k'^2-k^2)EK + \{8k'^2(k'^2-k^2) - 16k^2k'^2\}K^2\right].$$

Taking this between 0 and 1, and evaluating the limit of the right-hand side, we get, without much difficulty,

$$24 - 16\int_0^1 (\log 2kk')K^2k(k'^2-k^2)\,dk = 24\log 2,$$

i.e. $$2\int_0^1 (\log 2kk')K^2k(k'^2-k^2)\,dk = 3(1-\log 2) \quad \ldots(52).$$

While, if we differentiate (40) twice before putting n zero, we get

$$-24I_0' - 12\left\{\frac{\partial}{\partial n} I_n'\right\}_{n=0} - \left\{\frac{\partial^2}{\partial n^2} I_n'\right\}_{n=0}$$
$$= [2\log(2kk')]^2\left[4E^2 - 8k'^2EK + (4k'^2 - 8k^2k'^2)K^2\right]$$
$$+ 4\log(2kk')\left[-8(k'^2-k^2)EK + \{8k'^2(k'^2-k^2) - 16k^2k'^2\}K^2\right]$$
$$+ (16 - 64k^2k'^2)K^2.$$

Taking this between 0 and 1, and evaluating the limit of the right-hand side, we get

$$96 - 12.8\,(3 - 3\log 2) - 8\int_0^1 \{2\log(2kk')\}^2 K^2k\,(k'^2 - k^2)\,dk$$
$$= 144\,(\log 2)^2 - 4\pi^2.$$

That is to say

$$\int_0^1 \{2\log(2kk')\}^2 K^2k\,(k'^2 - k^2)\,dk$$
$$= \tfrac{1}{2}\pi^2 - 18\,(\log 2)^2 + 36\log 2 - 24 \ldots (53).$$

From (52) and (53) we can deduce some further integrals; from (40) we have

$$\int_0^1 (2kk')^{2\alpha+2n} K^2k\,(k'^2 - k^2)\,dk$$
$$= \left\{\frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{(\alpha+\frac{3}{2})(\alpha+\frac{5}{2})\cdots(\alpha+n+\frac{1}{2})}\right\}^3 \int_0^1 (2kk')^{2\alpha} K^2k\,(k'^2 - k^2)\,dk.$$

Therefore

$$\int_0^1 \sum_{n=0}^{\infty} \left\{\frac{(\alpha+\frac{1}{2})(\alpha+\frac{3}{2})\ldots(\alpha+n-\frac{1}{2})}{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}\right\}^3 (2kk')^{2\alpha+2n} K^2k\,(k'^2 - k^2)\,dk$$
$$= \sum_{n=0}^{\infty} \left\{\frac{\alpha+\frac{1}{2}}{\alpha+n+\frac{1}{2}}\right\}^3 \int_0^1 (2kk')^{2\alpha} K^2k\,(k'^2 - k^2)\,dk.$$

If now we put $\alpha = 0$, first in this equation as it stands; second, after differentiating it with regard to α; and, third, after differentiating it twice with regard to α; we get, using the notation of § 4 and writing

$$\frac{1}{1^m} + \frac{1}{3^m} + \frac{1}{5^m} + \ldots = \tau_m,$$

the three results

$$\int_0^1 z_1 K^2k\,(k'^2 - k^2)\,dk = \tau_3 \int_0^1 K^2k\,(k'^2 - k^2)\,dk,$$

$$\int_0^1 z_2 K^2k\,(k'^2 - k^2)\,dk = (6\tau_3 - 6\tau_4)\int_0^1 K^2k\,(k'^2 - k^2)\,dk$$
$$+ \tau_3 \int_0^1 2\log(2kk')\,K^2k\,(k'^2 - k^2)\,dk,$$

$$\int_0^1 z_3 K^2k\,(k'^2 - k^2)\,dk = (24\tau_3 - 72\tau_4 + 48\tau_5)\int_0^1 K^2k\,(k'^2 - k^2)\,dk$$
$$+ 2\,(6\tau_3 - 6\tau_4)\int_0^1 2\log(2kk')\,K^2k\,(k'^2 - k^2)\,dk$$
$$+ \tau_3 \int_0^1 \{2\log(2kk')\}^2 K^2k\,(k'^2 - k^2)\,dk.$$

Rewriting these, we get, without much difficulty from (52) and (53)

$$\int_0^{1/\sqrt{2}} \frac{4}{\pi^2} K^4 k\,(k'^2 - k^2)dk + \int_{1/\sqrt{2}}^1 \frac{4}{\pi^2} K^2 K'^2 k\,(k'^2 - k^2)\,dk = -\tfrac{1}{2}\tau_3 \quad \ldots\ldots\ldots(54),$$

$$-\int_0^1 \frac{4}{\pi} K^3 K' k\,(k'^2 - k^2)\,dk = (6\tau_3 - 6\tau_4)\,(-\tfrac{1}{2}) + \tau_3 3\,(1 - \log 2) - 6\log 2\,(-\tfrac{1}{2}\tau_3)$$

$$= 3\tau_4,$$

$$\int_0^{1/\sqrt{2}} 4K^2 K'^2 k\,(k'^2 - k^2)\,dk + \int_{1/\sqrt{2}}^1 4K^4 k\,(k'^2 - k^2)\,dk$$

$$= (24\tau_3 - 72\tau_4 + 48\tau_5)\,(-\tfrac{1}{2})$$
$$+ 2\,(6\tau_3 - 6\tau_4)\,3\,(1 - \log 2)$$
$$+ \tau_3\{\tfrac{1}{2}\pi^2 - 18\,(\log 2)^2 + 36\log 2 - 24\}$$
$$- 12\log 2\,\{(6\tau_3 - 6\tau_4)\,(-\tfrac{1}{2}) + \tau_3 3\,(1 - \log 2)\}$$
$$+ 36\,(\log 2)^2\,(-\tfrac{1}{2}\tau_3)$$
$$= (\tfrac{1}{2}\pi^2)\,\tau_3 - 24\tau_5 \quad \ldots\ldots\ldots(55).$$

Now it is not difficult to see that

$$\int_0^1 K^2 K'^2 k\,(k'^2 - k^2)\,dk = 0,$$

by changing the independent variable to k', and then interchanging k and k'.

And consequently from (54) and (55)

$$\int_0^1 K^4 k\,(k^2 - k'^2)\,dk = 6\tau_5 \quad \ldots\ldots\ldots(56)$$
$$= 6\left\{\frac{1}{1^5} + \frac{1}{3^5} + \frac{1}{5^5} + \ldots\right\}$$
$$= \int_0^1 K'^4 k\,(k'^2 - k^2)\,dk$$

by interchanging k, k'.

While, as has been shown,

$$\int_0^1 K^3 K' k\,(k^2 - k'^2)\,dk = (\tfrac{3}{4}\pi)\,\tau_4 \quad \ldots\ldots\ldots(57)$$
$$= \frac{1}{128}\pi^5$$
$$= \int_0^1 K K'^3 k\,(k'^2 - k^2)\,dk,$$

by interchanging k, k'.

An interesting result, given by integrating the formula

$$\frac{d}{dk}\{(kk')^n K^{2n}\} = nk(kk')^{n-2} K^{2n-1}(K-2E),$$

is $$\int_0^1 (kk')^{n-2} K^{2n-1}(K-2E)\,kdk = 0 \ldots\ldots\ldots\ldots (58),$$

where n is any quantity greater than zero; and, in particular,

$$\int_0^1 K^4 kdk = 2\int_0^1 K^3 Ekdk \ldots\ldots\ldots\ldots\ldots (59).$$

It is possible to evaluate a large number of integrals which are of a somewhat similar nature to those given by (56) and (57), but which involve E or E' in the integrand.

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ON THE PROPAGATION OF WAVES IN AN ATMOSPHERE OF VARYING DENSITY.

By J. H. C. Searle, B.Sc., University College, London.

On the propagation of Waves in an Atmosphere.

1. IT is customary in considering the propagation of waves in the atmosphere to regard the latter as a homogeneous and isothermal medium. That the atmosphere is neither homogeneous nor isothermal there is abundant evidence to show, and, since we are concerned almost exclusively with the air as the vehicle for the transmission of sound waves and barometric disturbances, it becomes of some

interest to examine how the heterogeneity and non-uniformity of temperature influence the character of the disturbances.

Such disturbances are of two kinds:—(*a*) long waves, for which Boyle's law as an equation connecting pressure and density is a sufficiently good approximation, and (*b*) sound waves, for which the adiabatic relation must be used. Good examples of the former type are met with in the periodic alterations of the barometer at various points of the earth's surface. It will not be necessary to perform two separate investigations for the above two cases, since the former is a particular case of the latter, obtained by putting the ratio of the specific heats equal to unity. It will be sufficient, therefore, in most cases to deduce the former from the latter by putting $\gamma = 1$.

Plane Waves.

Suppose the density to vary as some function of x, but that in all planes perpendicular to the x-axis the density is constant at any instant. The axis of x will afterwards be taken vertical in considering the earth's atmosphere.

Write density

$$= \Delta = \rho\,(1 + s) \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where s represents the variations of density due to the sound waves, and ρ represents the "static" density in the undisturbed medium.

Thus, s is a function of x and t, ρ is a function of x alone.

We have

$$\frac{d\Delta}{dt} + \frac{d\,(\Delta u)}{dx} = 0.$$

Therefore

$$\rho\,\frac{ds}{dt} + \rho\,(1 + s)\,\frac{du}{dx} + u\rho\,\frac{ds}{dx} + u\,\frac{d\rho}{dx}\,(1 + s) = 0.$$

Therefore, neglecting small terms like the squares of velocities,

$$\frac{ds}{dt} + \frac{du}{dx} + u\,\frac{d\log\rho}{dx} = 0 \ldots\ldots\ldots\ldots\ldots (2).$$

Also

$$\frac{1}{\Delta}\,\frac{dp}{dx} = X - \frac{du}{dt} - u\,\frac{du}{dx},$$

where X is now retained as the cause of the variation of density along the x-axis.

Therefore, with the same approximations,

$$\frac{1}{\rho}\frac{dp}{dx} = X - \frac{du}{dt} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Taking $$p = \kappa\Delta^\gamma = \kappa\rho^\gamma(1+\gamma s) \quad \ldots\ldots\ldots\ldots\ldots (4),$$

therefore $$\frac{dp}{dx} = \kappa\gamma\rho^{\gamma-1}\frac{d\rho}{dx} + \kappa\gamma\rho^\gamma\frac{ds}{dx},$$

therefore $$\frac{1}{\Delta}\frac{dp}{dx} = \kappa\gamma\left[\rho^{\gamma-2}\frac{d\rho}{dx} + \rho^{\gamma-1}\frac{ds}{dx}\right] = X - \frac{du}{dt} \quad \ldots\ldots (5).$$

Differentiate with regard to the time. Therefore, since ρ is not a function of the time,

$$\kappa\gamma\rho^{\gamma-1}\frac{d^2s}{dxdt} = \frac{dX}{dt} - \frac{d^2u}{dt^2};$$

or, if X be a function of x alone,

$$\kappa\gamma\rho^{\gamma-1}\frac{d^2s}{dxdt} = -\frac{d^2u}{dt^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Differentiate (2) with regard to x: therefore

$$\frac{d^2s}{dxdt} + \frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d\log\rho}{dx} + u\frac{d^2\log\rho}{dx^2} = 0 \quad \ldots\ldots (7).$$

Therefore, eliminating $\frac{d^2s}{dxdt}$ between (6) and (7),

$$\frac{d^2u}{dt^2} = \kappa\gamma\rho^{\gamma-1}\left[\frac{d^2u}{dx^2} + \frac{du}{dx}\frac{d\log\rho}{dx} + u\frac{d^2\log\rho}{dx^2}\right] \ldots (8).$$

If ρ be constant and equal to ρ_0, we reach the ordinary equation

$$\frac{d^2u}{dt^2} = \kappa\gamma\rho_0^{\gamma-1}\frac{d^2u}{dx^2},$$

or $$\frac{d^2\phi}{dt^2} = a^2\frac{d^2\phi}{dx^2}.$$

To integrate (8) we require ρ as a function of x.

3. Consider the equilibrium of a column of air of unit cross section at a height z above the surface of the earth.

Let r_0 be the earth's radius, g the value of gravity at its surface.

We have

$$\rho\delta z g\left(\frac{r_0}{r_0+z}\right)^2+(p+\delta p)-p=0.$$

therefore $$\rho g\left(\frac{r_0}{r_0+z}\right)^2=-\frac{dp}{dz},$$

or $$\rho g\left(1-2\frac{z}{r_0}\right)=-\frac{dp}{dz}.$$

Also $$p=RT\rho \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where $R=$ gas constant,

$T=$ absolute temp.

Therefore $$\rho g\left(1-2\frac{z}{r_0}\right)=-R\frac{d(\rho T)}{dz},$$

therefore $$\frac{d(\rho T)}{\rho T}=-\frac{n}{T}\left(1-2\frac{z}{r_0}\right)dz,$$

therefore $$\rho=\frac{A}{T}e^{-n\int\{1-2(z/r_0)\}\,dz/T} \quad\ldots\ldots\ldots\ldots\ldots(10),$$

where $n=\dfrac{g}{R}$, and A is an arbitrary constant.

Therefore when T is known as a function of z, ρ is also known as a function of z.

4. In what follows we shall neglect the curvature of the earth and the variation of gravity; (10) becomes

$$\rho=\frac{A}{T}e^{-n\int dx/T} \quad\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Therefore $$\log\rho=\log A-\log T-n\int dx/T,$$

$$\frac{d}{dx}\log\rho=-\frac{1}{T}\frac{dT}{dx}-\frac{n}{T}=-\frac{1}{T}\left(n+\frac{dT}{dx}\right),$$

$$\frac{d^2}{dx^2}\log\rho=\frac{1}{T^2}\frac{dT}{dx}\left(n+\frac{dT}{dx}\right)-\frac{1}{T}\frac{d^2T}{dx^2}.$$

Putting in (8)

$$u=U\cos pt,$$

we have

$$\frac{d^2U}{dx^2} + \frac{dU}{dx}\frac{d\log\rho}{dx} + U\left[\frac{d^2\log\rho}{dx} + c^2\rho^{1-\gamma}\right] = 0 \ldots (12).$$

Or, substituting from (11),

$$\frac{d^2U}{dx^2} - \frac{1}{T}\left(n + \frac{dT}{dx}\right)\frac{dU}{dx}$$

$$+ U\left[\frac{1}{T^2}\frac{dT}{dx}\left(n + \frac{dT}{dx}\right) - \frac{1}{T}\frac{dT}{dx} + b^2 T^{\gamma-1} e^{n(\gamma-1)\int dx/T}\right] = 0 \ldots (13),$$

where $$b^2 = c^2 A^{1-\gamma},$$

and $$v^2 = \frac{p^2}{\kappa\gamma}.$$

(13) is the equation giving the particle velocity when the temperature is known as a function of the height. We shall proceed to consider a few of the special cases of temperature distribution which have been used in correcting for astronomical refraction. Before doing so, however, it must be remarked that any attempt to allow for temperature effects must always be exceedingly arbitrary and unreliable, at any rate within two or three miles of the surface of the earth. Conduction, radiation, and convection from land, sea, and mountain; the presence of water vapour and clouds; the prevailing winds; all these conspire to produce a state of the atmosphere which must be anything but constant from moment to moment. Indeed, the temperature gradient often undergoes a complete inversion between mid-day and midnight at the same place. Thus the most that can be done is to take a wide average, specify the time of day, and allow a very generous margin for errors due to the above effects. The almost imperceptible motion of the cirrus clouds would tend to show that in its higher reaches the atmosphere may be in a more quiescent state, a result which is borne out by the recent balloon and kite observations at Blue Hill Observatory, Hyde Park, Mass. At heights of four miles or more it would, therefore, be possible to formulate a temperature law with more exactness than is possible near the earth's surface.

The above considerations show that any law of temperature variation would be more or less tentative, and the very fact that the phenomenon is so complicated shows that much is to be gained by simplicity in any proposed law.

With this proviso we may return to the discussion of the differential equation (13). We consider first

Newton's Law.

5. Putting $T = \text{const.} = T_0$, (11) becomes

$$\rho = \rho_0 e^{-(n/T_0)x} = \rho_0 e^{-(g/RT_0)x} \quad \ldots\ldots\ldots\ldots (14),$$

therefore

$$\left.\begin{aligned} \log\rho &= \log\rho_0 - \frac{g}{RT_0}x \\ \frac{d}{dx}\log\rho &= -\frac{g}{RT_0} \\ \frac{d^2}{dx^2}\log\rho &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (15);$$

therefore, substituting in (12),

$$\frac{d^2U}{dx^2} - \kappa_1\frac{dU}{dx} + c_1^2 U e^{ax} = 0 \quad \ldots\ldots\ldots\ldots (16),$$

where $$\kappa_1 = \frac{g}{RT_0}, \quad c_1^2 = c^2\rho_0^{1-\gamma} = \left(\frac{2\pi}{\lambda_0}\right)^2 \quad \ldots\ldots\ldots\ldots (17)$$

$$a = \frac{g}{RT_0}(\gamma - 1),$$

λ_0 = wave length of sound at earth's surface.

By the successive substitutions

$$e^{ax} = \xi, \quad U = \xi^{\kappa_1/2a}V, \quad \text{and} \quad \left(\frac{c_1}{a}\right)^2\xi = \eta^2 \quad \ldots\ldots (18),$$

equation (16) becomes

$$\eta^2\frac{d^2V}{d\eta^2} + \eta\frac{dV}{d\eta} + \left[4\eta^2 - \left(\frac{\kappa_1}{a}\right)^2\right]V = 0 \ldots\ldots (19),$$

therefore $$V = AJ_{\kappa_1/a}(2\eta) + BJ_{-\kappa_1/a}(2\eta),$$

or $$U = e^{\frac{1}{2}\kappa_1 x}\left[AJ_{\kappa_1/a}\left(2\frac{c_1}{a}e^{\frac{1}{2}ax}\right) + BJ_{-\kappa_1/a}\left(2\frac{c_1}{a}e^{\frac{1}{2}ax}\right)\right],$$

or $$u = e^{\frac{1}{2}\kappa_1 x}\left[AJ_{1/(\gamma-1)}\left(2\frac{c_1}{a}e^{\frac{1}{2}ax}\right) + BJ_{1/(1-\gamma)}\left(2\frac{c_1}{a}e^{\frac{1}{2}ax}\right)\right]\cos pt$$

$$\ldots\ldots (20).$$

One of the constants A or B may be eliminated by making the surface of the earth a node. Thus $u=0$ when $x=0$; or

$$AJ_{1/(\gamma-1)}\left(2\frac{c_1}{a}\right)+BJ_{1/(1-\gamma)}\left(2\frac{c_1}{a}\right)=0 \quad \text{.....(21).}$$

The remaining constant depends on the mode and amplitude of excitation of the disturbance.

Ivory's Law.

6. There is a formula associated with the name of Ivory which makes the temperature a linear function of the height. The law, as given by Lord Kelvin* is

$$T=T_0\left[1-\frac{\gamma-1}{V_0^2}gx\right] \quad \text{.....(22),}$$

but we may consider the more general case

$$T=T_0-\beta x \quad \text{.....(23),}$$

from which the special formula above may be obtained by suitably choosing β. (11) now gives

$$\rho=\rho_0(1-\epsilon x)^m \quad \text{.....(24),}$$

where
$$m=\frac{n}{\beta}-1=\frac{g}{R\beta}-1$$

and
$$\epsilon=\frac{\beta}{T_0}$$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad \text{.....(25).}$$

Substituting the above value of ρ in the differential equation (8), we find

$$\frac{d^2U}{dx^2}-\frac{m\epsilon}{1-\epsilon x}\frac{dU}{dx}+\left[c^2(1-\epsilon x)^s-\frac{m\epsilon^2}{(1-\epsilon x)^2}\right]U=0,$$

where
$$c^2=\frac{p^2\rho_0^{1-\gamma}}{\kappa\gamma}$$

and
$$s=m(1-\gamma)$$

$$\left.\begin{array}{l} \\ \\ \end{array}\right\} \quad \text{.....(26).}$$

In the equation above, make the successive substitutions

$$(1-\epsilon x)^{s+2}=v^2 \quad \text{and} \quad U=v^{(1-m)/(s+2)}z.$$

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* *Manchester Memoirs*, 1862.

We find

$$v^2\frac{d^2z}{dv^2}+v\frac{dz}{dv}+z\left[\left(\frac{2cv}{\epsilon t}\right)^2-\sigma^2\right]=0\ldots\ldots\ldots(27),$$

where $\sigma=\dfrac{m+1}{s+2}=\dfrac{n\beta}{(n-\beta)(1-\gamma)+2\beta}$; $t=s+2$.

Thus, we have

$$z=AJ_\sigma\left(\frac{2c}{\epsilon t}v\right)+BJ_{-\sigma}\left(\frac{2c}{\epsilon t}v\right),$$

or

$$U=(1-\epsilon x)^{\frac{1}{2}(1-m)}\left[AJ_\sigma\left(\frac{2c}{\epsilon t}(1-\epsilon x)^{\frac{1}{2}t}\right)+BJ_{-\sigma}\left(\frac{2c}{\epsilon t}(1-\epsilon x)^{\frac{1}{2}t}\right)\right]\ldots(28),$$

and the particle velocity $=u=U\cos pt$.

Again making the earth a node, we have

$$AJ_\sigma\left(\frac{2c}{\epsilon t}\right)+BJ_{-\sigma}\left(\frac{2c}{\epsilon t}\right)=0\ldots\ldots\ldots\ldots(29).$$

In the special case of temperature gradient given by (22) we put $\beta=\dfrac{\gamma-1}{V_0^2}gT_0$ and remember that $RT_0=\dfrac{V_0^2}{\gamma}$.

The equation for u then becomes

$$u=(1-\epsilon x)^b\left[AJ_\sigma\{\mu\sqrt{(1-\epsilon x)}\}+BJ_{-\sigma}\{\mu\sqrt{(1-\epsilon x)}\}\right]\cos pt\ldots(30),$$

where
$$\left.\begin{aligned}&\epsilon=\frac{\gamma-1}{V_0^2}g, \qquad \mu=\frac{2pV_0}{g(\gamma-1)}\\ &b=\frac{2-\gamma}{2(\gamma-1)}, \qquad \sigma=\frac{\gamma}{\gamma-1}\end{aligned}\right\}\ldots\ldots\ldots(31).$$

The relation between A and B is now

$$AJ_\sigma(\mu)+BJ_{-\sigma}(\mu)=0.$$

7. Before entering into numerical considerations of (20), (28), and (30), we may investigate what happens in the case of long undulations for which γ is closely equal to unity, (17) then gives $a=0$ and (16) becomes

$$\frac{d^2U}{dx^2}-\kappa_1\frac{dU}{dx}+c_1^2U=0\ldots\ldots\ldots\ldots(32).$$

Thus $$U = Ae^{\lambda_1 x} + Be^{\lambda_2 x},$$

where $$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = \tfrac{1}{2}\left[\kappa_1 \pm \sqrt{(\kappa_1^2 - 4c_1^2)}\right] = \mu + \nu, \text{ say.}$$

Therefore $$u = (Ae^{\lambda_1 x} + Be^{\lambda_2 x})\cos pt,$$

or, if $u = 0$ when $x = 0$, we have

$$u = Ae^{\mu x}\sinh \nu x \cos pt \ldots \ldots\ldots\ldots(33),$$

where

$$\mu = \tfrac{1}{2}\frac{g\gamma}{V_0^2},\ \nu = \tfrac{1}{2}\sqrt{(\kappa_1^2 - 4c_1^2)} = \tfrac{1}{2}\sqrt{\left\{\frac{g^2\gamma^2}{V_0^4} - \frac{16\pi^2}{\lambda_0^2}\right\}}.$$

An equation of this form has already been given by Lord Rayleigh.*

In the case of Ivory's law, if we at once put $\gamma = 1$, then, remembering that $\beta = \dfrac{\gamma - 1}{V_0^2}\, g T_0$, (24) and (25) give immediately

$$\begin{aligned}\rho &= \rho_0 (1 - \epsilon x)^m \\ &= \rho_0 \left(1 - \frac{\gamma - 1}{V_0^2}\, T_0 g x\right)^{1/(\gamma - 1)} \\ &= \rho_0 \left(1 - \frac{\gamma - 1}{\gamma}\, \frac{g}{R}\, x\right)^{1/(\gamma - 1)} \\ &= \rho_0 e^{-(g/R)x}, \text{ since } \gamma = 1.\end{aligned}$$

Thus, we return to Newton's law and thence to (33).

Let us take a general value for β which does not vanish when $\gamma = 1$. Then, $m \neq \infty$ when $\gamma = 1$, but $s = 0$ by (26).

This leads to

$$(1 - \epsilon x)^2 \frac{d^2 U}{dx^2} - m\epsilon (1 - \epsilon x)\frac{dU}{dx} - m\epsilon^2 U = 0 \ldots (34),$$

which gives

$$U = A(1 - \epsilon x) + B(1 - \epsilon x)^{-m},$$

therefore $$u = A\left[(1 - \epsilon x) - (1 - \epsilon x)^{-m}\right]\cos pt \ldots\ldots(35).$$

If we now make $\gamma = 1$, this becomes

$$u = A\left\{1 - e^{(g/V_0^2)x}\right\}\cos pt \ldots\ldots\ldots\ldots(36).$$

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* *Phil. Mag.*, No. 29, 1890, p. 173.

8. It is at once evident from (31) that in the case of long waves the velocity increases indefinitely with the height. It would be difficult to say what would happen in the case of (20) and (30) without first entering into numerical values. Consider (17) and (20): we find

$$\kappa_1 = 1.176 \times 10^{-6};\ c_1 = 5.5 \times 10^{-2};\ a = 4.68 \times 10^{-7};\ 2\frac{c_1}{a} = 2.2 \times 10^{5}$$

expressed in *c.g.s.* units.

Also

$$u = e^{(\frac{1}{2}\kappa_1)x}\left[AJ_{1/(\gamma-1)}\left(2\frac{c_1}{a}e^{\frac{1}{2}ax}\right) + BJ_{1/(1-\gamma)}\left(2\frac{c_1}{a}e^{\frac{1}{2}ax}\right)\right]\cos pt.$$

It appears that the arguments of the Bessel Functions are large and we may therefore use the asymptotic expansion for J_n, viz.,

$$J_n = \sqrt{\left(\frac{2}{\pi z}\right)}\left[1 - \frac{(1^2-4n^2)\,(3^2-4n^2)}{2!\,(8z)^2} + \ldots\right]\cos\{z - \tfrac{1}{4}\pi - n(\tfrac{1}{2}\pi)\}$$

$$+ \sqrt{\left(\frac{2}{\pi z}\right)}\left[\frac{1^2-4n^2}{1.8z} - \frac{(1^2-4n^2)\,(3^2-4n^2)\,(5^2-4n^2)}{3!\,(8z)^3} + \ldots\right]$$

$$\times \sin\{z - \tfrac{1}{4}\pi - n\,(\tfrac{1}{2}\pi)\},$$

true when the real part of n is $> -\frac{1}{2}$.

Now $$J_{1/(\gamma-1)} = J_{2.5},\ \text{closely},$$

but $$J_{1/(1-\gamma)} = J_{-2.5},$$

and thus the above expansion is not valid for $J_{-2.5}$. We may, however, bring it within reach of the expansion by means of the recurrence formula

$$J_{n-1} = \frac{2n}{z}J_n - J_{n+1},$$

true for all values of n and z.

By repeated use of this formula, we find

$$J_{-2.5} = \left(\frac{3}{z^3} + \frac{2}{z}\right)J_{.5} + \left(1 - \frac{3}{z^2}\right)J_{1.5}.$$

Or, remembering that z is great,

$$J_{-2.5} = J_{1.5},\ q.p.$$

By the above expansion for $J_n(z)$, when z is great, we have

$$J_{2.5} = \sqrt{\left(\frac{2}{\pi z}\right)}(-\sin z),$$

and

$$J_{1.5} = \sqrt{\left(\frac{2}{\pi z}\right)}(-\cos z).$$

Thus

$$u = e^{(\frac{1}{2}\kappa_1)x}\sqrt{\left(\frac{2}{\pi z}\right)}(-A\sin z - B\cos z)\cos pt,$$

very approximately, where $z = 2\dfrac{c_1}{a}e^{\frac{1}{2}ax}$,

or

$$u = e^{\frac{1}{4}\kappa_1(3-\gamma)x}\sqrt{\left(\frac{a\lambda_0}{2\pi}\right)}(A_1\sin z + B_1\cos z)\cos pt.$$

Thus u increases indefinitely with x since $(A_1\sin z + B_1\cos z)$ has a maximum value $\sqrt{(A_1^2 + B_1^2)}$.

The exponential factor increases to twice its primitive value when $e^{\frac{1}{4}\kappa_1(3-\gamma)x} = 2$, or when $x = 14$ kilometres, so that the effect of the variation of density is small.

9. Similar considerations in the case of Ivory's law leads to the result

$$u = \sqrt{\left(\frac{2}{\pi\mu}\right)}\sqrt{(1-\epsilon x)}\,[A\cos z - B\sin z]\cos pt,$$

where

$$z = \mu\sqrt{(1-\epsilon x)},$$

and

$$\epsilon = 3.34 \times 10^{-7},$$

$$\mu = 3.3 \times 10^{4},$$

while in (27)

$$b = \tfrac{3}{4}.$$

The effect of the elevation of temperature in ascending is now to diminish the particle velocity which thus stands in contrast to the case previously considered.

The factor $\sqrt{(1-\epsilon x)}$ decreases to one-half of its original value when

$$\sqrt{(1-\epsilon x)} = \tfrac{1}{2},$$

or $$4 - 4\epsilon x = 1,$$

$$x = \frac{3}{4\epsilon} = \tfrac{3}{4} \times \tfrac{1}{3} \times 10^{+7} = \tfrac{1}{4} \times 10^{+7}$$

$$= 25 \text{ kilometres, } q.p,$$

so that the effect is the reverse of the case previously considered and of a still smaller order.

In reality, however, it matters very little whether the velocity increase or decrease as we ascend in the air. In the former case a state of affairs is ultimately reached at which our equations cease to represent the motion, since the hypothesis of small motions becomes untenable. In the latter case the low temperatures reached at greater heights would facilitate the deposition of moisture in the form of clouds which would materially alter our hypotheses.

What we have to see to specially is that the energy of the motion remains finite. In the case of Ivory's law this is evident, since the velocity decreases uniformly from a finite value to zero. In the case of Newton's law we have

$$\text{energy of motion} = \tfrac{1}{2}\int_0^\infty \rho u^2 dx$$

$$= \tfrac{1}{2}\int_0^\infty [AJ_n + BJ_{-n}]^2 dx,$$

the exponential factor vanishing, since it is involved in both ρ and u. Now the argument of the integral is always finite and the integral is convergent at the upper limit. Thus the energy of the whole motion is strictly finite.

10. We shall now discard the restriction to one-dimensional propagation and investigate the vibrations of an isothermal atmosphere in which the "static" density varies according to any law in a vertical direction, but is uniform over all horizontal strata.

Refer the motion to cylindrical coordinates r, ϑ, z—z being vertical.

The equation of continuity is given by

$$r\frac{d\Delta}{dt} + \frac{d}{dr}(ur\Delta) + r\frac{d}{dz}(v\Delta) + \frac{d}{d\vartheta}(\omega\Delta) = 0.$$

In the case of a disturbance originating at a pole (the origin) the motion is symmetrical about the z-axis, therefore

$$r\frac{d\Delta}{dt} + \frac{d}{dr}(ur\Delta) + r\frac{d}{dz}(v\Delta) = 0 \ldots\ldots\ldots(37).$$

Putting in this

$$\Delta = \rho\,(1+s),$$

where ρ is a function of z only, and s is a function of z and t, we find, neglecting squares of velocities, &c.,

$$\rho\frac{ds}{dt}+\rho\frac{du}{dr}+\rho\frac{dv}{dz}+v\frac{d\rho}{dz}=0,$$

or

$$\frac{ds}{dt}+\frac{du}{dr}+\frac{dv}{dz}+v\frac{d\log\rho}{dz}=0 \quad \ldots\ldots\ldots\ldots(38).$$

The equations of motion in cylindrical coordinates are

$$\left.\begin{aligned} r\Delta\frac{Du}{dt} &= rR\Delta-\frac{d}{dr}(rp)\\ \text{and}\qquad \Delta\frac{Dv}{dt} &= Z\Delta-\frac{dp}{dz}\end{aligned}\right\}\ldots\ldots\ldots\ldots(39),$$

R and Z being body forces and p the pressure.

In the case of the earth's atmosphere

$$R=0,\quad Z=g,$$

$$\left.\begin{aligned} \text{therefore}\qquad r\frac{du}{dt} &= -\frac{p}{\Delta}-\frac{r}{\Delta}\frac{dp}{dr}\\ \text{and}\qquad \frac{dv}{dt} &= -\frac{1}{\Delta}\frac{dp}{dz}+g\end{aligned}\right\}\ldots\ldots\ldots\ldots(40).$$

But

$$\begin{aligned} p &= \kappa\Delta^{\gamma}\\ &= \kappa\rho^{\gamma}(1+\gamma s);\end{aligned}$$

therefore

$$\frac{p}{\Delta}=\kappa\rho^{\gamma-1},$$

$$\frac{dp}{dr}=\kappa\gamma\rho^{\gamma}\frac{ds}{dr},$$

$$\frac{dp}{dz}=\kappa\gamma\rho^{\gamma-1}\frac{d\rho}{dz}+\kappa\gamma\rho^{\gamma}\frac{ds}{dz},$$

since ρ is independent of r, but not of z; therefore

$$\left.\begin{aligned} \frac{du}{dt} &= -\frac{\kappa}{r}\rho^{\gamma-1}-\kappa\gamma\rho^{\gamma-1}\frac{ds}{dr}\\ \frac{dv}{dt} &= -\kappa\gamma\rho^{\gamma-2}\frac{d\rho}{dz}-\kappa\gamma\rho^{\gamma-1}\frac{ds}{dz}+g\end{aligned}\right\}\ldots\ldots\ldots\ldots(41).$$

We have to determine u and v from (38) and (41) by eliminating s.

From (41)

$$\frac{d^2u}{dt^2} = -\kappa\gamma\rho^{\gamma-1}\frac{d^2s}{drdt}.$$

From (38)

$$\frac{d^2s}{drdt} + \frac{d^2u}{dr^2} + \frac{d^2v}{drdz} + \frac{dv}{dr}\frac{d\log\rho}{dz} = 0,$$

therefore

$$\frac{d^2u}{dt^2} = \kappa\gamma\rho^{\gamma-1}\left[\frac{d^2u}{dr^2} + \frac{d^2v}{drdz} + \frac{dv}{dr}\frac{d\log\rho}{dz}\right] \ldots\ldots(42).$$

But $$v = \frac{d\phi}{dz},$$

therefore $$\frac{dv}{dr} = \frac{d^2\phi}{drdz} = \frac{du}{dz},$$

therefore $$\frac{d^2u}{dt^2} = \kappa\gamma\rho^{\gamma-1}\left[\frac{d^2u}{dr^2} + \frac{d^2u}{dz^2} + \frac{du}{dz}\frac{d\log\rho}{dz}\right] \ldots\ldots(43).$$

Again, from (41),

$$\frac{d^2v}{dt^2} = -\kappa\gamma\rho^{\gamma-1}\frac{d^2s}{dzdt},$$

and, from (38),

$$\frac{d^2s}{dzdt} + \frac{d^2u}{drdz} + \frac{d^2v}{dz^2} + \frac{dv}{dz}\frac{d\log\rho}{dz} + v\frac{d^2\log\rho}{dz^2} = 0,$$

therefore

$$\frac{d^2v}{dt^2} = \kappa\gamma\rho^{\gamma-1}\left[\frac{d^2u}{drdz} + \frac{d^2v}{dz^2} + \frac{dv}{dz}\frac{d\log\rho}{dz} + v\frac{d^2\log\rho}{dz^2}\right] \ldots(44).$$

But $$u = \frac{d\phi}{dr},$$

therefore $$\frac{du}{dz} = \frac{d^2\phi}{dzdr} = \frac{dv}{dr},$$

therefore

$$\frac{d^2v}{dt^2} = \kappa\gamma\rho^{\gamma-1}\left[\frac{d^2v}{dr^2} + \frac{d^2v}{dz^2} + \frac{dv}{dz}\frac{d\log\rho}{dz} + v\frac{d^2\log\rho}{dz^2}\right] \ldots(45).$$

(43) and (45) determine u and v in terms of r, z, and t for a heterogeneous atmosphere in which the density varies in plane layers according to any law.

11. We may apply these results to the case of the earth's atmosphere, assuming the earth to be an indefinite plane.

Newton's law is

$$\rho = \rho_0 e^{-(g/c)z},$$

where ρ = density at surface,

c = const. in Boyle's equation

$$p = c\rho.$$

Using this relation, we have

$$\left.\begin{aligned} \frac{d^2u}{dt^2} &= \kappa\gamma\rho_0{}^{\gamma-1}e^{-m(\gamma-1)z}\left[\frac{d^2u}{dr^2}+\frac{d^2u}{dz^2}-m\frac{du}{dz}\right] \\ \frac{d^2v}{dt^2} &= \kappa\gamma\rho_0{}^{\gamma-1}e^{-m(\gamma-1)z}\left[\frac{d^2v}{dr^2}+\frac{d^2v}{dz^2}-m\frac{dv}{dz}\right] \end{aligned}\right\} \ldots(46),$$

where $$m = \frac{g}{c},$$

showing that to our approximation and with this law of variation of density u and v are the same functions of r, z, and t.

Putting

$$\left.\begin{aligned} u &= U\cos pt \\ \kappa\gamma\rho_0{}^{\gamma-1} &= a^2 \\ m(\gamma-1) &= b \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(47),$$

and, substituting, we have

$$\frac{d^2U}{dr^2}+\frac{d^2U}{dz^2}-m\frac{dU}{dz}+\frac{p^2}{a^2}e^{bz}U=0 \ldots\ldots(48).$$

Again, putting

$$U = Z\cos qr \ldots\ldots\ldots\ldots\ldots\ldots(49),$$

therefore

$$\frac{d^2Z}{dz^2}-m\frac{dZ}{dz}+\left(\frac{p^2}{a^2}e^{bz}-q^2\right)Z=0\ldots\ldots\ldots(50).$$

In this make the successive substitutions

$$u = \frac{p}{a}e^{\frac{1}{2}bz} \text{ and } Z = u^{m,b}v = u^{1/(\gamma-1)}v\ldots\ldots\ldots(51).$$

There results

$$u^2\frac{d^2v}{du^2}+u\frac{dv}{du}+v\left[\frac{4}{b^2}u^2-\mu^2\right]=0\ldots\ldots\ldots(52),$$

from which

$$v=AJ_\mu\left(\frac{2u}{b}\right)+BJ_{-\mu}\left(\frac{2u}{b}\right),$$

therefore

$$Z=\left(\frac{2\pi}{\lambda_0}\right)^{1/(\gamma-1)}e^{(g/2c)z}\left[AJ_\mu\left(\frac{4\pi}{\lambda_0 b}e^{\frac{1}{2}bz}\right)+BJ_{-\mu}\left(\frac{4\pi}{\lambda_0 b}e^{\frac{1}{2}bz}\right)\right],$$

or, finally, the velocity u (or v) is given by

$$u=v=e^{(g/2c)z}\left[CJ_\mu(\alpha e^{\frac{1}{2}bz})+DJ_{-\mu}(\alpha e^{\frac{1}{2}bz})\right]\cos qr.\cos pt\ldots(53),$$

where

$$\left.\begin{aligned}
&\mu^2=\frac{g^2+4q^2c^2}{g^2(\gamma-1)^2}\\
&\alpha=\frac{4\pi}{\lambda_0 b}\\
&b=\frac{g}{c}(\gamma-1)\\
&p=\frac{2\pi}{T}\text{ and }T=\text{time period}\\
&q=\frac{2\pi}{\lambda}\text{ and }\lambda=\text{horizontal wave length}\\
&c=\text{const. in Boyle's equation }p=c\rho\\
&g=\text{acc. of gravity (assumed const.)}\\
&\gamma=\text{ratio of specific heats for air}
\end{aligned}\right\}\ldots\ldots(54).$$

The assumption of Ivory's law of variation of temperature would lead to results somewhat similar to those previously obtained and need not be worked out in detail.

It is clear that the variations of density at the surface of the earth have a vanishingly small effect on the amplitudes of aerial disturbances. In the case of a smaller planet the effect would be greater, but the variation of gravity and curvature of surface would then enter as complicating factors. As it remains, therefore, the problem derives interest from an analytical point of view only.

I have to thank Prof. Karl Pearson for kindly reading over this paper and for making many helpful suggestions.

ON THE DISTINCTION OF RIGHT AND LEFT AT POINTS OF DISCONTINUITY.

By W. H. Young, Sc.D., F.R.S.

§ 1. THAT a discontinuous function of a single real variable may at a given point behave differently in its right-hand and in its left-hand neighbourhoods is well-known. For a function to have particular properties it is sometimes necessary that its behaviour in some definite particular should be the same on both sides at every point at which the function is defined. Sometimes again, on the other hand, it is sufficient that the behaviour of the function at the point on one of the two sides at least should be of a certain character.

We may illustrate the first case by the necessary and sufficient condition* that a discontinuous function which is the sum of a series of continuous functions should have the property of assuming in every interval every value between its upper and lower limits in that interval (and therefore, in particular, assuming in every interval every value between the values at the extremities of that interval). The condition is that the value of the discontinuous function at every point should be a limit of values assumed by the function in the neighbourhood of the point on the left and on the right also.

As an example of the second case, I may mention the fact that for the sum of a series of functions to be pointwise discontinuous with respect to every perfect set, it is sufficient that the functions should be everywhere continuous on one side at least.†

In the present paper I propose to examine what the possibilities are as regards the distinction of right and left, and to show, in particular, that the most general conceivable

* It should be noted that this condition is easily secured, and that we are thus led to infer the existence of a large class of functions which possess the property, often erroneously given as the characteristic property, of a continuous function. Young, "A theorem in the theory of functions of a real variable," 1907, *Rend. di Palermo*, XXIV.

† Young, "A new proof of a theorem of Baire's," 1907, *Messenger of Math.*, Vol. XXXVII., pp. 49–54.

function of a single variable can differ as regards right and left at only a countable number of points, which can, however, be everywhere dense. I may add that I have obtained similar results with regard to the distinction of right and left at points of non-uniform convergence of a series of continuous functions. These results I hope to publish shortly in another paper.*

§ 2. Let P be any internal point of a segment throughout which a function $f(x)$ is defined. Take any interval with P as right-hand end-point (fig. 1), then $f(x)$ has, for the points internal to this interval, an upper limit; as this interval diminishes, this upper limit cannot increase, and therefore has a limit, which is, at the same time, its lower limit; denote this limit by $\phi_L(P)$.

We thus get for every point P of the segment a function $\phi_L(x)$, or shortly ϕ_L, which may be called *the upper left-hand limiting function of* $f(x)$.

Similarly, changing left into right, we define a function ϕ_R, *the upper right-hand limiting function of* f. Further, interchanging the words "upper" and "lower," "increase" and "decrease," in the definition, we define corresponding *lower limiting functions* which we shall denote by ψ_L and ψ_R.

If at each point P we choose that one of the two upper limiting functions which is not less than the other, we get a new function, which may be called the (modified) *associated upper limiting function*, and be denoted by ϕ. Similarly we define the (modified) *lower limiting function* ψ, by taking that one of the two lower limiting functions which is not less than the other.

The word "modified" has been inserted in these definitions to distinguish the functions in question from the corresponding functions extensively used by Baire†, and elsewhere by myself‡, and which I shall now distinguish as ϕ_B and ψ_B. The word "modified" will be dropped, and, for brevity, the associated functions will usually be denoted by their symbols.

It is obvious that $\phi_B = \phi$, *except where* $f > \phi$.

Throughout the following investigations theorems are usually only enunciated and proved for the ϕ's, the corresponding theorems for the ψ's can, as is well known, be always obtained by changing the sign of f.

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* A short account of these results, as well as of those of the present paper, was given by me at the meeting of the British Association at Leicester, August 5, 1907.

† Baire. *Sur les fonctions de variables réelles*, 1899, *Ann. di Mat.*, Ser. 3, III., pp. 1–123. ϕ_B is that one of ϕ and f which is not less than the other.

‡ Young. *The general Theory of Integration*, 1904, *Phil. Trans.* A., CCIV., p. 232, seq. *Upper and Lower Integration*, 1904, *Proc. L.M.S.*, 2, II, p. 55, seq.

Some interesting facts which can be expressed in terms of these functions are known, *e.g.* Baire's theorem that ϕ_B *is an upper semi-continuous function, and, as such, at most pointwise discontinuous*, and Pasch's* theorem that *if the points at which* $\phi_R - \psi_R$ (or $\phi_L - \psi_L$) *are* $> k$ *form a set of zero content,* $f(x)$ *is integrable.*

No complete discussion of these functions appears however to exist. In particular, although Pasch's theorem, quoted above, suggests the possibility of the functions not being independent, their mutual relations have not been investigated. The theorems of the present paper will, I believe, be found to clear up the subject; in the light of them for instance, Pasch's theorem and certain other kindred theorems† become obvious.

The greater number of the results of this paper refer to the distinction of right and left with respect to the associated limiting functions. Previous work on discontinuous functions has either entirely ignored this distinction (*e.g.* Baire's Thesis above referred to) or has emphasized it in the statement of definitions and the like, and, in particular in the classification of discontinuities, without discussing how far the facts of the case warranted this. The examples commonly given of discontinuous functions have always been those where the distinction of right and left either did not exist, or played a trivial part.‡

§ 3. In connection with Baire's theorem, mentioned above, it may be pointed out that the word "continuous" must be understood in the extended sense in which a function is said to be continuous at P, provided, as x approaches P as limit in any manner, $f(x)$ approaches always a single limit, and that limit is $f(P)$. It is easily seen that this definition is the same as the following:—

A function f *is said to be continuous at* P *if the five associated numbers at* P *are all equal, viz.*

$$f(P) = \phi_L(P) = \phi_R(P) = \psi_L(P) = \psi_R(P).$$

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* Pasch. *Math. Ann.*, XXX., p 141.

† Dini, *Funzioni di Variabili Reali*, § 187; Schoenflies *Bericht*, p 139.

‡ Mark the importance of testing not only the *accuracy* but also the *scope* of one's results by constructing examples. To quote an instance which has come under my notice in the course of my present work, Dini (p. 307) states that if a left-hand derivate and a right-hand derivate both exist and are finite and different at every point of an interval.....certain results follow.

The reader might well imagine not only that such a case could occur, but that Dini knew of a case where it did occur. As a matter of fact, however, the hypthesis is an impossible one. In default of an example it could, in such a case, only stimulate research to state that an example had not been found.

This definition includes the ordinary definition in all cases when $f(x)$ is bounded*, but admits of $f(x)$ being always finite but not bounded, or of $f(x)$ being itself infinite.

It is most necessary to bear this distinction in mind when dealing with the associated functions, as is brought out by the following simple example in which f is always finite, but the associated functions always infinite.

Ex. 1. Let

$f=$ 0 at every irrational point;

$f=$ q at every rational point p/q, where q is even, and

$f=-q$ when q is odd (p being less than q and prime to it).

Here $$\phi_L=\phi_R=+\infty,$$

$$\psi_L=\psi_R=-\infty.$$

§ 4. It appears from the above that the exclusion of proper infinite values in a paper dealing with the associated functions would be an anomaly. Where, however, this exclusion introduces a material simplification in the proof as well as imparts an extended character to the result, it has been made, the general theorem being at the same time stated and proved (§§ 12–14). Unless expressly stated, therefore, it is to be understood that infinite values are allowed.†

§ 5. Since, by a theorem of Baire's, the points at which a function which is upper semi-continuous has a value $\leq k$, for fixed k, form a closed set, it follows that *the points where* $\phi \geq k$ *form a closed set.* Similarly, *the points, if any, where* $\phi=+\infty$ *form a closed set;* the same is true of the points where $\psi \leq k$ or $=-\infty$. If the function is pointwise discontinuous and finite, the sets $\phi=+\infty$, $\psi=-\infty$ are dense nowhere; without this restriction they may, as in Ex. 1, fill up the whole continuum.

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* Borné, limited (Hobson)—that is when the upper and lower limits of f in any region considered are finite. For obvious reasons I have used the term "bounded" instead of "limited;" the word limit would otherwise occur in too many connections.

† In Baire's Thesis, although he uses the expression "fonction quelconque" the functions considered must be taken to be bounded. Thus Baire gives and constantly applies the theorem "ϕ_D-f has its lower limit zero in every domain," which, as in Ex. 1, is not true if f is not bounded. Baire's theorem, quoted above, has been extended to the case when f is not bounded, and I have myself lately published a proof of it without restriction *Mess. of Math., loc. cit.*

The following theorem gives us corresponding information about the points at which ϕ or any other upper semi-continuous function $= -\infty$.

The set of the points, if there are any, at which an upper semi-continuous function $\phi = -\infty$, is an ordinary inner limiting set.*†

For, if P be a point at which $\phi = -\infty$, we can surround P by an interval at every internal point of which $Q < -n$, n being any chosen integer. Let d be the largest such interval with P as centre, and let the set of intervals d for all positions of P be denoted by D_n.

The inner limiting set E of the sets $D_1, D_2, \ldots, D_n \ldots$ then certainly contains the points $\phi = -\infty$; further, it contains no other points; for, if Q be any point at which $\phi \neq -\infty$, we can assign an integer k such that $\phi(Q) > -k$, so that Q is certainly not internal to any of the intervals D_k, and therefore does not belong to the inner limiting set E. Q. E. D.

§ 6. Baire has shown that the function ϕ_B is upper semi-continuous; the following theorem, which will be found in the sequel of constant application, shows that a corresponding statement may be made with respect to each of the associated functions.

THEOREM 1. *Any limit approached by $\phi(x)$, $\phi_L(x)$, $\phi_R(x)$, or $\phi_B(x)$ as x approaches a point P as limit on the right $\leq \phi_L(P)$, and as x approaches P as limit on the left $\leq \phi_R(P)$.*

For, by the definition of ϕ_L, we can find an interval d (fig. 2) with P as right-hand end-point, such that at every point x internal to d

$$f(x) \leq \phi_L(P) + e,$$

e being any positive quantity previously selected.

Thus, describing any interval δ with x as internal or end-point, and sufficiently small to lie entirely inside d, we shall have for every point inside δ, including or excluding the point x,

$$f(\xi) \leq \phi_L(P) + e.$$

Hence by their definitions each of the functions $\phi(x)$, $\phi_L(x)$, $\phi_R(x)$ is $\leq \phi_L(P) + e$.

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* For an example of a case where there are such points see Ex. 6.

† Hence the set has the potency c, unless it contains no component dense in itself, when it is countable. Young, *Theory of Sets of Points*, p. 64 (Cambridge University Press, 1906).

Since e is at our disposal, this proves the first part of the theorem; similarly, approaching P from the right, we get the second part of the theorem.

Cor. 1. *ϕ_L is upper semi-continuous on the left* and ϕ_R on the right, while ϕ, like ϕ_B, is an upper semi-continuous function, and, as such, at most pointwise discontinuous.*

Cor. 2. *At any point where*

$$\phi_L = \phi_R = \phi,$$

both ϕ_L and ϕ_R are upper semi-continuous (on both sides).

It is hardly necessary to point out that the converse of Cor. 2 does not hold; the following example shows this.

Ex. 2. Let

$$f\left(-\frac{1}{2^n}\right) = 1, \quad f\left(+\frac{1}{2^n}\right) = 2,$$

and elsewhere $f = 0$.

Then, except at the origin,

$$\phi_L(x) = \phi_R(x) = 0,$$

while at the origin

$$\phi_L(0) = 1, \quad \phi_R(0) = 2,$$

and both ϕ_L and ϕ_R are upper semi-continuous.

§ 7. Functions, such as ϕ_L and ϕ_R, which may be said to be *one-sided upper semi-continuous*, have properties strictly analogous to those proved by Baire (pp. 8, 11, 12) for symmetrically upper semi-continuous functions. Thus:—

If f is a function which is upper semi-continuous on the right, the points at which $f \geq k$ or $f = +\infty$, is closed on the left.†

If a function f is upper semi-continuous on the right and has its lower limit zero everywhere, the points $f = 0$ are dense everywhere.

If such a function f is always positive, there is in every interval an interval in which the lower limit of f is positive.

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* That is to say $\phi_L \geq$ its own associated left-hand upper limiting function.

† That is to say, includes all its limiting points except possibly such as are limiting points on the right only, that is right-hand end-points of black intervals of the set.

The proofs given by Baire require little modification; proofs, however (such as those used by Baire in proving other properties of upper semi-continuous functions), which involve the property that f and ϕ_B have the same maximum in any interval, cannot be transferred; it is easily seen that this property does not hold for f and ϕ_L, or for f and ϕ_R, or indeed for f and ϕ.

All these theorems, however, gain in interest after proving Theorem 5.

§ 8. THEOREM 2. *At every point of continuity of* ϕ,

$$\phi_L = \phi_R = \phi,$$

and both ϕ_L *and* ϕ_R *are continuous.*

For, as x approaches a point P at which ϕ is continuous, whether x approaches from the left or the right, $\phi(x)$ has the limit $\phi(P)$.

Therefore, by Theorem 1, $\phi(P)$ is not greater than either $\phi_L(P)$ or $\phi_R(P)$, so that, by the definition of ϕ,

$$\phi_L(P) = \phi_R(P) = \phi(P).$$

This proves the first statement.

Again, since ϕ is continuous at P, we can assign an interval d with P as middle point (fig. 3) such that, if x is any point of d except P,

$$\phi(x) \geq k,$$

where k is any quantity less than $\phi(P)$ (or, if $\phi(P) = -\infty$, is also $= -\infty$).

Therefore, if x_1 be any point of d other than P, any limit approached by $\phi(x)$ as x approaches x_1 will be $\geq k$. Therefore

$$\phi_L(x_1) \geq k, \text{ and also } \phi_R(x_1) \geq k.$$

Thus any limit approached by either ϕ_L or ϕ_R, as x_1 approaches P, $\geq k$, and therefore, k being at our disposal, provided only it is $< \phi(P)$, any such limit $\geq \phi(P)$.

But, by Theorem 1, no such limit is greater than $\phi(P)$, therefore it is always $= \phi(P)$.

Since, as has been shown,

$$\phi(P) = \phi_L(P) = \phi_R(P),$$

this shows that both ϕ_L and ϕ_R are continuous at P. Q. E. D.

Cor. ϕ_L *and* ϕ_R, *as well as* ϕ, *are at most pointwise discontinuous.*

(N.B. ϕ_L or ϕ_R may have points of continuity other than those of ϕ.)

§ 9. The reasoning used in the proof of Theorem 2 is unaltered if, instead of ϕ, we use ϕ_B. Hence we have the following theorem:—

Theorem 3. *The points of continuity of* ϕ_B *are among the points of continuity of* ϕ, *so that at each*

$$\phi_B = \phi = \phi_L = \phi_R \geq f;$$

at any point of continuity of ϕ *which is not a point of continuity of* ϕ_B

$$\phi_B = f > \phi.$$

(N.B. That ϕ may have points of continuity other than those of ϕ_B is shown by the following example; it is shown in § 14 that such points, if any, are countable.)

Ex. 3. Let $f = 0$, except at one or more isolated points where $f = 1$.

This function is upper semi-continuous and therefore $= \phi_B$; it is discontinuous wherever its value is 1; ϕ, however, is 0 always, and therefore continuous.

§ 10. Theorem 4. *The only points at which both* ϕ_L *and* ϕ_R *are continuous are the points of continuity of* ϕ.

For, if P be a common point of continuity of ϕ_L and ϕ_R, as x approaches P from the left, $\phi_R(x)$ approaches as limit $\phi_R(P)$; therefore, by Theorem 1,

$$\phi_R(P) \leq \phi_L(P).$$

Similarly, approaching P from the right,

$$\phi_L(P) \leq \phi_R(P).$$

Therefore $\phi_L(P) = \phi_R(P) = \phi(P)$.

Now, if we take any sequence with P as limiting point, passing along which $\phi(x)$ has a definite limit, at each point ϕ is the same as either ϕ_L or ϕ_R, and therefore we can so pick out the points of the sequence that ϕ is always the same as one of these functions, say ϕ_L; the limit will therefore be

$\phi_L(P)$, that is, as has been shown $\phi(P)$; thus ϕ is continuous at P. Q.E D.

§ 11. Results exactly analogous to those obtained for the ϕ's hold, of course, for the ψ's, lower semi-continuity replacing upper semi-continuity.

The points of discontinuity of a pointwise discontinuous function form an ordinary outer limiting set of the first category, that is the outer limiting set of a series of closed sets dense nowhere and each contained in the succeeding. It at once follows that the discontinuities of two, or any finite number of, pointwise discontinuous functions also form an ordinary outer limiting set of the first category.

The complementary set of a set of the first category with respect to any segment is, as is shown by Baire, a set of the second category, dense everywhere, of potency c, and having common points with every set of the second category, these common points forming, as is easily proved, themselves a set of the second category. Such a set is therefore formed by the common points of continuity of ϕ and ψ, points at which, as has been shown, ϕ, ϕ_L, and ϕ_R are all equal, and so are ψ, ψ_L, ψ_R, so that we may say there is no distinction of right and left. Thus we get the following statement:—

There is no distinction of right and left except possibly at points of an ordinary outer limiting set of the first category.

The following example shows, however, that the exceptional points may be dense everywhere. It will be noticed that in the example the exceptional points are countable; this suggests, what is in fact true, that the above statement does not give us all the information possible about the exceptional points. It is shown in § 12 that *such points, if any, are always countable.* Thus the following example may be taken as typical of a function having the most complicated possible distribution of points at which there is a distinction of right and left.

Ex. 4. Divide the segment (0, 1) of the x-axis by continued bisection so as to construct in order the terminating binary fractional points $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, ...

Divide the segment (0, 1) of the y-axis by continued trisection and blackening of the middle segment, so as to get the black intervals of Cantor's typical ternary perfect set (fig. 4).

Let each binary point as it is constructed in order on the x-axis correspond to the middle point of the black interval constructed at the same stage on the y-axis.

If we then let each limiting point on the x-axis correspond to the limiting point of the corresponding points on the y-axis, the correspondence would give us a monotone function, pointwise discontinuous, and, within the limits of vision, represented in fig. 4.

If, however, instead, we assign at each point other than the terminating binary fractional points, a value greater by unity than the value of the above monotone function, we get a totally discontinuous function having at every terminating binary fractional point all five numbers ϕ_L, ϕ_R, ψ_L, ψ_R, f, distinct, while, at the remaining points, there is no distinction of right or left.

Arithmetically, using the binary scale for x and the ternary scale for $y=f(x)$, we have, e_1 denoting a figure which may be either 0 or 1, and e_i' standing for $2e_i$,

$$f(0.e_1e_2\ldots e_n1) = 0.e_1'e_2'\ldots e_n'1,$$

$$f(0.e_1e_2\ldots e_ne_{n+1}\ldots) = 1.e_1'e_2'\ldots e_n'e'_{n+1}\ldots.$$

At the latter point we have

$$f=\phi_L=\phi_R=1+\psi_L=1+\psi_R;$$

but at the former point we have

$$\psi_L=0.e_1'e_2'\ldots e_n'1=\phi_L-1,$$

$$\psi_R=0.e_1'e_2'\ldots e_n'2=\phi_R-1,$$

so that all five numbers are distinct.

§ 12. We shall require the following Lemma from the Theory of Sets of Points, which is here stated, I believe, for the first time. It is closely analogous to Cantor's Theorem of the Nucleus.

Lemma. If S is a set which is more than countable, it contains a component U' which is dense in itself on both sides, and more than countable in every interval containing a point of U'.

Using the nomenclature of Young's *Theory of Sets of Points*, p. 53, we call a point of S an L'-point, if in every interval containing the point there is a more than countable set of points of S.

It is then known that a set is, or is not, countable, according as it has not, or has, an L'-point, and that, in the latter case, the set of L'-points is dense in itself and more than countable, while the remaining points of S are countable. It follows

that in the neighbourhood of any L'-point there is a more than countable set of L'-points.

Now, if P is an L'-point which is a limit of L'-points on its right, but not on its left, there is an interval with P as right-hand end-point containing no L'-point. Thus such one-sided L'-points, being end-points of a set of non-overlapping intervals, are, like the intervals, countable.

Next, let Q be an L'-point which is a limit on both sides of L'-points.

Then in any interval with Q as end-point there is an L'-point, and therefore a more than countable set of L'-points, and therefore, by what has just been proved, certainly there will be another L'-point which is a limit on both sides of L'-points. Thus such a point Q is a limit on both sides of points of the same kind.

Thus, if we remove the one-sided limits there remains over such a set U' as is mentioned in the enunciation. Q.E.D.

§ 13. THEOREM 5. *The points, if any, at which*

$$\phi_R > \phi_L \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

are countable.

For let k be any chosen number, and consider the set of points S at which

$$\phi_R > \phi_L + k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

By the sense of this equation ϕ_R is not $-\infty$, therefore the set S may be considered as the outer limiting set of the sets S_1, S_2, ..., where S_r is the set of points at which, beside (2),

$$\phi_R > r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Thus, if S_r is countable for all values of r, S is countable.

Let us now make k assume in succession the values of a sequence of positive quantities with zero as limit; the outer limiting set of the sets S will then be the whole set of points satisfying (1); this set will therefore be countable if S is so. Thus it is sufficient to show that S_r is countable to prove the theorem. This we can do by the Lemma.

For suppose, if possible, that S_r contained a component U dense in itself on both sides, and let P be any point of the set U.

By reason of (3) we then have

$$r < \phi_L(P),$$

as a result of Theorem 1. Therefore, by (2),

$$\phi_R(P) > r + k \quad \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Since this is true at every point of U, it may be taken to replace (3); therefore, by the same argument,

$$\phi_R > r + 2k,$$

and similarly

$$\phi_R > r + nk,$$

for all values of n, at every point of U. Hence

$$\phi_R = +\infty$$

at every point of U.

Hence, by Theorem 1, since there are certainly points of U on the left of P,

$$\phi_L(P) = +\infty,$$

which is clearly incompatible with (1) or (2).

Thus the assumption that S_r has such a component is untenable, so that, by the Lemma, S_r is countable, which proves the Theorem.

§ 14. The importance of this theorem justifies giving the shortened form of the proof in the case when the function f is bounded. At the same time we are able to give in this case a more complete enunciation.

Theorem 5a. *If f is a bounded function, the points, if any, at which*

$$\phi_R \geq \phi_L + k \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

form a set which has no component dense in itself on the left at least.

For if there were such a component U, and P were a point of it, we have by Theorem (1), near P on the left,

$$\mathrm{Lt}\, \phi_R(x) \leq \phi_L(P),$$

so that by (2) we can certainly find a point x of U at which

$$\phi_R(x) < \phi_R(P) - \tfrac{1}{2}k.$$

By a repetition of the argument, we get for any value of n a point x_n of U at which

$$\phi_R(x_n) \leqq \phi_R(P) - \tfrac{1}{2}nk,$$

which is clearly impossible if f and therefore ϕ_R is bounded.

That the set of points satisfying (2) may be dense in itself, provided it is so on the right alone, is shown in Ex. 5.

That the statement in Theorem 5a is not necessarily true when the function is unbounded is shown in Ex. 6. The restriction to bounded functions is however not necessary, it is indeed sufficient if the function f is always finite. The most general form of the theorem, suggested by Ex. 6, is that given below as Theorem 5b.

Ex. 5. Let the black intervals of a perfect set, dense nowhere, be (A_1, B_1), (A_2, B_2), (A_3, B_3), ... (fig. 4).

At every point of the perfect set let $f = 1$, while elsewhere $f = 0$. Here $\phi_L = \phi_R$, everywhere except at A_1, A_2, ... where $\phi_L = 1$, $\phi_R = 0$; and at B_1, B_2, ... where $\phi_R = 1$, $\phi_L = 0$.

Thus the set

$$\phi_R - \phi_L \geq k,$$

provided $k \leqq 1$, is dense in itself on the right, while

$$\phi_L - \phi_R \geq k$$

is dense in itself on the left. Both sets are, of course, countable.

Ex. 6. As in the preceding example let (A_1, B_1), (A_2, B_2), ... be the black intervals of a perfect set dense nowhere. At every point outside these intervals let $f = -\infty$.

At every point of the closed interval (A_n, B_n) let $f = -n$.

Then at every point where $f = -\infty$, $\phi = -\infty$ also, and we have a distinction of right and left at the end-points of the black intervals only.

Now at A_n we have $\phi_R = -n$, $\phi_L = -\infty$; while at B_n we have $\phi_L = -n$, $\phi_R = -\infty$. Thus the set of points

$$\phi_R - \phi_L \geq k$$

consists for all values of k of the points A_1, A_2, ... which is dense in itself on the left; while the set

$$\phi_L - \phi_R \geq k$$

consists of the points B_1, B_2, ... which is dense in itself on the right.

THEOREM 5b. *If there is a set of points dense in itself on the left at least at every point of which*

$$\phi_R - \phi_L \geq k \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

then the points at which

$$f = -\infty,$$

and at the same time

$$\phi = -\infty,$$

are more than countable.

For, if U be such a set of points, and P any point of U, we can show as in the proof of Theorem 5a, that, however large n may be, we can find a point x_n of U, as near as we please to P, at which

$$\phi_R(x_n) \leq \phi_L(P) - \tfrac{1}{2}(n-1)k.$$

Therefore (since by (2), $\phi_L(P)$ is not $+\infty$), we can find an integer n, such that for this and all subsequent integers,

$$\phi_R(x_n) \leq -M,$$

a fortiori, by (2)

$$\phi_L(x_n) \leq -M.$$

Therefore, by Theorem 1, we can find a whole interval d_1, having x_n as right-hand end-point, throughout which both f and ϕ are less than $-M$: since x_n is a point of U, there certainly are points of U internal to d_1.

Applying the same reasoning to this interval, we get another interval d_2 containing points of U, such that throughout it both f and ϕ are less than $-2M$.

Proceeding thus *ad infinitum* we determine at least one point L internal to all the intervals, at which

$$f(L) = -\infty,$$

and also

$$\phi(L) = -\infty.$$

Since there is at least one of these points L in the neighbourhood of every point P of U, they form a set, which, like U, is dense in itself. But, by construction, they form an inner limiting set, therefore* they are more than countable. Q.E.D.

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* *Theory of Sets of Points*, p. 64.

Ex. 7. Let us use the binary notation, by which, e_i being either 0 or 1,

$$.e_1, e_2, \ldots, e_n, \ldots = \tfrac{1}{2}e_1 + \tfrac{1}{4}e_2 + \ldots + \frac{e_n}{2^n} + \ldots.$$

At every non-terminating binary fractional point let $f = -\infty$.

Any terminating fraction may be expressed in the form

$$.e_1 e_2 \ldots e_n 0^r 1^s,$$

where 0^r stands for r consecutive zeros, and 1^s for s consecutive ones. At this point let

$$f = -2^{n+r}.$$

Then we have at the same point

$$\phi_L = -2^{n+r+s},$$

and

$$\phi_R = -\infty.$$

For as we approach the point from the left, the terminating binary fractions have the limit

$$.e_1 e_2 \ldots e_n 0^r 1^{s-1} 0\dot{1},$$

while the limit on the right is

$$.e_1 e_2 \ldots e_n 0^r 1^s \dot{0}.$$

Thus the set

$$\phi_L - \phi_R \geq k$$

consists for every value of k, of the terminating binary fractional points; that is the set is countable but dense everywhere, and therefore, of course, dense in itself on both sides.

§ 15. Any one of the Theorems 5, 5a, or 5b shows that the points, if any, where $\phi_R > \phi_L$ are countable, and, of course similarly, those where $\phi_L > \phi_R$ are countable. *Thus the points, if any, where ϕ_R differs from ϕ_L are countable.* Combining this with the corresponding results for the lower limiting functions ψ_R and ψ_L, it follows that *the points, if any, where there is a distinction of right and left are countable.* This completes the investigations of § 9.

§ 16. The reasoning used in proving Theorem 5, depending as it does simply on Theorem 1, remains true if we change ϕ_R into f. Hence we have the following result:—

THEOREM 6. *The points, if any, at which*

$$f > \phi_L,$$

or $$f < \phi_R,$$

or $$f > \phi,$$

are countable.

Combining this with Theorem 5,

$$\phi_L = \phi_R = \phi \geq f \text{ and also } \psi_L = \psi_R = \psi \leq f,$$

at all but a countable set of points.

The more detailed results given in § 12, viz. Theorems 5a and 5b, have in the same way their exact counterparts here, using f instead of ϕ_R.

EX. 8. Let $f\left(\frac{p}{q}\right) = -q$, and $f(x) = -\infty$ at each irrational point x. The set $f > \phi$ consists of the rational points and is therefore dense everywhere.

EX. 9. The following examples involve only bounded functions, the set of points at which $f > \phi$ is dense everywhere.

(*a*) Let $f\left(\frac{p}{q}\right) = 1 + \frac{1}{q}$, while at each irrational point x, $f(x) = 0$.

Here $$\phi_L = \phi_R = \phi = 1, \ \psi_L = \psi_R = \psi = 0.$$

The points at which $f > \phi$ are the rational points.

(*b*) In the above the set of points at which $f > \phi + k$ is finite; by a slight modification of the values at the rational points we can make it infinite without altering ϕ or ψ.

Let $$f\left(\frac{p}{q}\right) = 1 + \frac{1}{q - p}.$$

Here, provided only $k < 1$, we can, to each integer q find at least one integer p prime to q and such that

$$1 \leq q - p < \frac{1}{k}.$$

At each such point $\frac{p}{q}$ we shall have $f > \phi + k$; thus this set is countably infinite.

(*c*) Both these functions are totally discontinuous. We may change them into pointwise discontinuous functions by assigning at the irrational points the value 1 instead of 0. This does not affect ϕ, but it makes $\psi = 1$, so that every irrational point is a point of continuity of f. The sets of points $f \geq \phi + k$ are unaffected by the change.

In contrast to the examples just given the following is an example of *a bounded function which is less than* ϕ *and greater than* ψ *everywhere.*

Ex. $$f\left(\frac{p}{2q}\right) = 2 - \frac{1}{q}, \; f\left(\frac{p}{2q+1}\right) = \frac{1}{q},$$

$f(x) = 1$, when x is irrational.

Here $$\phi = 2, \; \psi = 0.$$

§ 16. Thus we see that, although from the point of view of distribution in the continuum, the distinction of right and left may exist in every interval, however small, yet from the point of view of potency it is the exception, and not the rule, for there to be a distinction of right and left.

This is rather upsetting to the accepted classification of the discontinuities of a function of a single variable, in which a discontinuity is said to be of the first kind, or ordinary, if both the limits $f(x-0)$, $f(x+0)$ exist* and have definite values which differ from one another.†

We have, in fact, now learned that such "ordinary" discontinuities are at most countable, while the remaining discontinuities, which may be more than countable, must be properly considered as normal.

Again, if neither of the limits $f(x-0), f(x+0)$ exist, the discontinuity is said to be of the second kind.‡ Here we notice that the normal thing is to have three numbers f, ϕ, and ψ at such a discontinuity, and only at most in a countable set of cases can there be more; the oscillations on the right and left§ coincide therefore in the normal case.

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* That is $\phi_L = \psi_L = f(x-0)$ and $\phi_R = \psi_R = f(x+0)$.

† See Hobson, *Functions of a real variable*, 1907, § 181, p. 234.

‡ *Loc. cit.*, p. 235, lines 9–13.

§ *Loc. cit.*, p. 235, lines 14, 15.

Göttingen,
June, 1907.

ON A CURIOUS DYNAMICAL PROPERTY OF PARTICLES IN EQUILIBRIUM, AND ON SOME PROPERTIES OF SPHERICAL TRILINEAR COORDINATES.

By the late Dr. E. J. Routh.*

IN a short memoir published in the *Bulletin de la Societé Mathematique de France*, 1897, vol. XXV., M. Andrade has discussed a curious question concerning the stability of a material point. He showed that a particle in stable equilibrium under the separate actions of two systems of non-conservative forces may not be in stable equilibrium under the two systems acting simultaneously. In the next volume of the same journal M. Lecornu carries the proposition a step further, and shows that the mere introduction of a constraint without the addition of any active forces may be sufficient to make the equilibrium unstable.

These theorems, though not altogether new, seemed so interesting that I was led to examine the problem over again in a more general manner. As the method here employed is different from that of M. Lecornu, the theorems arrived at, though they do not contradict his results, are not common to the two discussions.

1. The position of equilibrium, O, being taken as the origin, let (xyz) be the coordinates of the moving particle referred to rectangular axes fixed in space. The forces may then be represented by

$$\left.\begin{aligned} X &= ax + fy + ez \\ Y &= f'x + by + dz \\ Z &= e'x + d'y + cz \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

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* This paper was written by Dr. Routh shortly before his death, and seems to have been left by him in a nearly complete state.

Mrs. Routh informs me that he was unable to work after the middle of April when his strength failed rapidly and he lost the power of concentration. He died June 7, 1907. On the first sheet of the paper in Dr. Routh's writing is the title "On a curious dynamical property of particles in equilibrium, arts. 1–10. Also some properties of spherical trilinear coordinates which appear to be new, arts. 11–19. [For the *Quarterly Journal of Mathematics*]."

The whole MS. is carefully written in ink in Dr. Routh's neat writing, so well known to his pupils, with a few pencil additions. It has been printed verbatim as it stands in the original MS., except for a few trifling alterations indicated by Routh himself in pencil, and due to his having changed the sign of p^2 from positive to negative in equation (2).—Editor.

The possible directions of rectilinear motion are then given by

$$\frac{X}{x} = \frac{Y}{y} = \frac{Z}{z} = -p^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which leads to

$$\left.\begin{aligned} (a+p^2)\,x + fy + ez &= 0 \\ f'x + (b+p^2)\,y + dz &= 0 \\ e'x + d'y + (c+p^2)\,z &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By eliminating the ratios of x, y, z we arrive at *a discriminating cubic* by which we can determine p^2.

2. It is therefore obvious that two cases arise: (1) that in which the cubic has three real roots which we represent by p_1^2, p_2^2, p_3^2, and (2) that in which the cubic has only one real root.

3. Let OA, OB, OC be the principal axes of the free motion; OL, OM, ON normals to the planes BOC, COA, AOB. Then on a sphere, centre O; L, M, N are the corners of *the polar triangle of* ABC. We shall suppose that the positive directions of OA, OB, OC are so chosen that the arcs AB, BC, CA on the sphere are each less than π. The spherical triangle ABC may be called *the principal triangle.*

Let $(x'y'z')$ be the *oblique coordinates* of the moving point P referred to the axes OA, OB, OC, and let $(X'\,Y'Z')$ be the oblique components of force. Then, since (Y', Z'), (Z', X') (X', Y') vanish when (y', z'), (z', x'), (x', y') are respectively zero, we have

$$X' = -p_1^2 x', \quad Y' = -p_2^2 y', \quad Z' = -p_3^2 z'.$$

Let (xyz) be the *rectangular coordinates* of P referred to any fixed axes, O being the origin. Let the direction cosines of OA, &c., OL, &c. referred to these axes be given by the diagrams

	x	y	z
A	α	β	γ
B	α'	β'	γ'
C	α''	β''	γ''

	x	y	z
L	λ	μ	ν
M	λ'	μ'	ν'
N	λ''	μ''	ν''

where we may replace the letters A, B, C by their equivalents x', y', z' if this is more convenient.

In the first case there are three real directions of possible rectilinear motion, which are not necessarily at right angles, and are sometimes called *the principal axes of the motion.* (See the author's *Dynamics of a Particle*, arts. 292, 513.) In the second case there is only one such axis. Let us consider these cases in order.

To fix our ideas we suppose the particle to be in *stable equilibrium* under the action of the system of forces (XYZ). In this case the three roots p_1^2, p_2^2, p_3^2 are positive, and the three periods of oscillation are $2\pi/p_1, 2\pi/p_2, 2\pi/p_3$. Let us now suppose that *a constraint is introduced into the system*, so that the particle is constrained to move on a surface which passes through the point O. *Our object is to determine whether the position of equilibrium at O remains stable or becomes unstable.*

4. Let (XYZ) be the components of force along the axes (xyz), then

$$-X = p_1^2 x'.\alpha + p_2^2 y'.\alpha' + p_3^2 z'.\alpha'',$$

$$-Y = p_1^2 x'.\beta + p_2^2 y'.\beta' + p_3^2 z'.\beta'',$$

$$-Z = p_1^2 x'.\gamma + p_2^2 y'.\gamma' + p_3^2 z'.\gamma''.$$

Resolve $(x'y'z')$ along OL, OM, ON, we find

$$x' \cos LA = x\lambda + y\mu + z\nu,$$

$$y' \cos MB = x\lambda' + y\mu' + z\nu',$$

$$z' \cos NC = x\lambda'' + y\mu'' + z\nu''.$$

Now, if we take the plane of xy as the tangent plane of the constraining surface, we have $z=0$. After eliminating $(x'y'z')$, we find

$$-X = \left\{p_1^2 \frac{\alpha\lambda}{\cos AL} + p_2^2 \frac{\alpha'\lambda'}{\cos BM} + p_3^2 \frac{\alpha''\lambda''}{\cos CN}\right\} x + \left\{p_1^2 \frac{\alpha\mu}{\cos AL} + p_2^2 \frac{\alpha'\mu'}{\cos BM} + p_3^2 \frac{\alpha''\mu''}{\cos CN}\right\} y,$$

$$-Y = \left\{p_1^2 \frac{\beta\lambda}{\cos AL} + p_2^2 \frac{\beta'\lambda'}{\cos BM} + p_3^2 \frac{\beta''\lambda''}{\cos CN}\right\} x + \left\{p_1^2 \frac{\beta\mu}{\cos AL} + p_2^2 \frac{\beta'\mu'}{\cos BM} + p_3^2 \frac{\beta''\mu''}{\cos CN}\right\} y.$$

This we may write shortly

$$-X=Ax+By, \quad -Y=B'x+Cy.$$

To solve these we write $-X=m^2x$, $-Y=m^2y$, and arrive at the equations

$$(A-m^2)x+By=0, \quad B'x+(C-m^2)y=0.$$

5. Conditions of stability. The quadratic giving the periods $(2\pi/m)$ of the constrained motion is

$$m^4-(A+C)m^2+(AC-BB')=0.$$

Put $\quad A+C=u, \ AC-BB'=v, \ u^2-4v=w.$

The roots will be real if w is positive, and both these roots will be positive if u and v are positive. *The conditions of stability* of the particle after it has been constrained to move on a fixed surface *are that all the three quantities u, v, w must be positive.*

6. The trilinear coordinates. The normal to the surface of constraint at O has been taken as the axis of z in the rectangular system of axes (art. 3). To prevent confusion with other systems of axes we designate this particular normal by the letters OQ, and its intersection with the sphere whose centre is O by the letter Q. We now put

$$\xi=\frac{\cos AQ.\cos LQ}{\cos AL}, \quad \eta=\frac{\cos BQ.\cos MQ}{\cos BM}, \quad \zeta=\frac{\cos CQ.\cos NQ}{\cos CN}.$$

These quantities may be regarded as the *spherical trilinear or areal coordinates of Q* when referred to either *the spherical triangle ABC or the polar triangle LMN.*

7. The curves u, v, w. We find by using the known properties of direction cosines given by the diagrams in art. 3, that

$$u=p_1^2(1-\xi)+p_2^2(1-\eta)+p_3^2(1-\zeta),$$

$$v=p_1^2p_2^2(1-\xi-\eta)+p_2^2p_3^2(1-\eta-\zeta)+p_3^2p_1^2(1-\zeta-\xi),$$

$$w=u^2-4v.$$

By turning the straight line OQ round O into any position defined by $(\xi\eta\zeta)$, *we determine at once, by observing the signs of uvw, whether the introduction of the corresponding constraint has introduced instability or not* (art. 5).

8. GEOMETRICAL FORMS. To put this result into a geometrical form, we refer the motion to a set of axes Ox, Oy, Oz, which are fixed while OQ moves. Let $(x_1y_1z_1)$ be the coordinates of any point in OQ, r_1 its distance from O; the direction cosines of OQ are then x_1/r_1, &c. Let the direction cosines of the fixed lines OA, OB, OC, OL, &c., be $(\alpha_1\beta_1\gamma_1)$, $(\alpha_1'\beta_1'\gamma_1')$, $(\alpha_1''\beta_1''\gamma_1'')$; $(\lambda_1\mu_1\nu_1)$, &c. (see art. 3). Then, by Art. 6,

$$r_1^2\xi = r_1^2 \cos AQ \cos LQ/\cos AL$$

$$= (\alpha_1x_1 + \beta_1y_1 + \gamma_1z_1)(\lambda_1x_1 + \mu_1y_1 + \nu_1z_1)/\cos AL.$$

Hence $r_1^2\xi$, and similarly $r_1^2\eta$, and $r_1^2\zeta$ are quadric functions of $(x_1y_1z_1)$. And the surfaces defined by $u=0$, $v=0$, $\xi=0$, $\eta=0$, $\zeta=0$ are quadric cones. *These cones separate the spaces on the sphere on one side of which u, v, ξ, &c., are respectively positive and on the other negative. The surface $w=0$ is a quartic cone possessing a similar property.*

9. When Q is at the corner C of the principal triangle (art. 3) $\xi=0$, $\eta=0$, $\zeta=1$, and therefore

$$u = p_1^2 + p_2^2, \quad v = p_1^2p_2^2, \quad w = (p_1^2 - p_2^2)^2.$$

When Q is at the corner N of the polar triangle we have again $\xi=0$, $\eta=0$, $\zeta=1$, and u, v, w have the same values as before. Thus u, v, w are positive at every corner of both triangles. When therefore *Q lies on the same side of the cone $u=0$ as any one of the six corners A, B, C, L, M, N, then u is positive at Q and one condition of stability is satisfied.* Similarly, when Q lies on the same side of the cone $v=0$ as any corner then v is positive and a second condition of stability is satisfied. In the same way, if a curve on the sphere can be drawn from any corner to the point Q which cuts the quartic cone $w=0$ in an even number of points, then w is positive at Q and the third condition of stability is satisfied.

We may therefore change the stability into instability by merely placing the normal OQ so that any one of these conditions is not satisfied.

10. The equations giving the values of ξ and u may be written in the forms

$$r_1^2\xi = f(x_1y_1z_1), \quad r_1^2(u - p_1^2 - p_2^2 - p_3^2) = F(x_1y_1z_1),$$

where $f(x_1y_1z_1)$ and $F(x_1y_1z_1)$ are quadric functions of $(x_1y_1z_1)$, whose coefficients are the same for all positions of the radius vector OQ. It follows that *the surfaces defined by any given values of* ξ *are concyclic and coaxial cones.* The same remark applies to the cones defined by any given values of η, ζ, u, v.

The circular sections of the cones defined by giving ξ any constant value are parallel to the planes

$$\alpha_1x_1+\beta_1y_1+\gamma_1z_1=0, \quad \lambda_1x_1+\mu_1y_1+\nu_1z_1=0,$$

and these planes are respectively at right angles to OA and OL (art. 8). Thus the two sets of circular sections of the cones ξ, η, ζ are perpendicular to (OA, OL), (OB, OM), (OC, ON) respectively.

11. On sperical trilinear coordinates. Let ABC be any spherical triangle, LMN its polar triangle. The spherical coordinates of a point P here used are

$$\xi=\frac{\cos AP.\cos LP}{\cos AL}, \quad \eta=\frac{\cos BP.\cos MP}{\cos BM}, \quad \zeta=\frac{\cos CP.\cos NP}{\cos CN}.$$

We may express these in terms of the elements of either triangle alone. Let AP, BP, CP produced cut the opposite sides of ABC in G, H, K (fig. 5), and let LP, MP, NP produced cut the sides of the same triangle in G', H', K'.

Since N is the pole of AB,

$$\cos NP=\sin PK'=\sin PK.\sin K,$$

$$\cos NC=\sin CF=\sin CK.\sin K.$$

Substitute in the expression for ζ, and the result contains the elements of the triangle ABC alone. We thus find

$$\xi=\frac{\cos AP.\sin GP}{\sin AG}, \quad \eta=\frac{\cos BP.\sin HP}{\sin BH}, \quad \zeta=\frac{\cos CP.\sin KP}{\sin CK}.$$

We have similar formulæ when the position of P is referred to the triangle LMN.

The arcs GP, HP, KP are taken positively when measured in the directions GP, HP, KP respectively. We notice that the arcs AL, BM, CN lie between $\pm\frac{1}{2}\pi$.

12. We shall now prove that *the invariable relation connecting these coordinates takes the simple form*

$$\xi+\eta+\zeta=1.$$

This relation is much simpler than that which holds between the perpendiculars from P on the sides when taken as the trilinear coordinates, and is also simpler than that between the angular distances AP, BP, CP. Both these last formulæ consist of seven terms each containing the product of four or five factors.

To prove this, we take OA, OB, OC as the axes of a system of oblique coordinates, $(x'y'z')$ being the coordinates of P. By projecting $r' = OP$ in succession on the normals OL, OM, ON to the planes BOC, COA, AOB (art. 3), we find

$$x' \cos AL = r' \cos LP, \quad y' \cos BM = r' \cos MP,$$

$$z' \cos CN = r' \cos NP.$$

Let Π be any other point on the sphere, then by projecting $r' = OP$ on $O\Pi$, we have

$$x' \cos A\Pi + y' \cos B\Pi + z' \cos C\Pi = r' \cos P\Pi.$$

Therefore

$$\cos P\Pi = \frac{\cos A\Pi \cdot \cos LP}{\cos AL} + \frac{\cos B\Pi \cdot \cos MP}{\cos BM} + \frac{\cos C\Pi \cdot \cos NP}{\cos CN}.$$

Similarly, interchanging P and Π,

$$\cos P\Pi = \frac{\cos AP \cdot \cos L\Pi}{\cos AL} + \frac{\cos BP \cdot \cos M\Pi}{\cos BM} + \frac{\cos CP \cdot \cos N\Pi}{\cos CN}.$$

Either of these formulæ gives the cosine of the arc joining any two points, and reduces to the ordinary expression

$$\cos P\Pi = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$$

when the axes OA, OB, OC are rectangular.

Let the points P, Π coincide, we then have

$$1 = \xi + \eta + \zeta.$$

The coordinate curves on the sphere defined by equating the coordinates ξ, η, ζ in succession to any constant may be called for brevity *the curves* ξ, η, ζ. The cones having these for base and the common vertex at O may be designated *the cones* ξ, η, ζ.

13. *To find the points of intersection of the quadric*

$$U = D\xi + E\eta + F\zeta = 0$$

with the sides of the triangle ABC, the coefficients *DEF* being constants. Let *P* be any such point, $\xi\eta$ its coordinates. Put $\zeta = 0$, and we have

$$D\xi + E\eta = 0, \quad \xi + \eta = 1 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Substitute from the second formulæ of art. 11; since *P* lies on *AB*, *G* coincides with *B*, and *H* with *A*, hence

$$D \tan BP + E \tan AP = 0 \ldots\ldots\ldots\ldots\ldots(2).$$

The quadric $U = 0$ therefore cuts any side *AB*, or *AB* produced, in points such that the ratio of the tangents of the segments is equal to the ratio $-D/E$. Since $AP + BP = c$, the equation (2) is practically a quadratic to find $\tan AP$ or $\tan BP$, and *supplies two values of* $\tan AP$ *real or imaginary.*

We may determine whether the intersections are real or imaginary more simply by solving the equations (1).

If the value of ξ thus found is inconsistent with the superior and inferior limitations to the values of ξ, the intersections are imaginary. To find these limitations we make $\xi \sin c$, that is, $\cos AP . \sin BP$ a max-min. The result shows that the maximum and minimum values of ξ are $\frac{1}{2}(\operatorname{cosec} c \pm 1)$, and that they occur at points distant $\pm\frac{1}{2}\pi$ from the middle point of *AB*.

14. Before proceeding further it will be convenient *to trace briefly the different positions of the coordinate curves* ζ. The polar equation of this curve, when referred to the middle point *T* of *CN* as origin, is obviously

$$\cos^2 n . \cos^2 \rho - \sin^2 n . \sin^2 \rho . \cos^2 \phi = \zeta \cos 2n,$$

where $CN = 2n$, $TP = \rho$, and the angle $CTP = \phi$, and ζ has any constant value.

For any given value of ϕ the polar equation leads to but one value of $\cos 2\rho$. It follows that points, the sum of whose distances from *T* is equal to π and which have the same ϕ, have also the same value of ζ. These two points are clearly equidistant, on opposite sides, from the great circle whose pole is *T*.

Let the arc *CN*, when produced, cut *AB*, *LM* in *F*, *F'*, then *NCFF'* is at right angles to both *AB* and *LM*. Let the sides *AB*, *LM* intersect in *I* and *J*. Then *I* and *J* are the poles of *NCFF'* and the polar circle of *T* passes through *I* and *J* and bisects *FF'* in some point *T'*.

The great circle IT'J divides the spherical surface into two equal parts and the values of ζ *are the same on each side.*

15. Since $\zeta = \cos CP . \cos NP / \cos 2n$, it follows that *when ζ is negative P lies between the two great circles AB, LM,* and ζ is a negative maximum when $\rho = \frac{1}{2}\pi$, $\phi = 0$, that is at the point T' (art. 13). Its value at T' is $-\sin^2 n / \cos 2n$ and the curve ζ is then reduced to the solitary point T'.

As ζ approaches zero (but is still negative) the ζ curves become elongated ovals which surround T' as their centre and extend from $\phi = -\phi_1$ to $\phi = +\phi_1$ where

$$\cos^2 \phi_1 = -\zeta \cos 2n / \sin^2 n.$$

When $\zeta = 0$, the curve ζ becomes the two sides AB, LM, and the cone ζ reduces to the two planes containing those sides. *Thus $\zeta = 0$ when P lies on any side of either of the triangles ABC, LMN.*

When ζ becomes positive, the curves encircle the point T (fig. 6). When $\zeta = 1$ the equation of the curve becomes $\cos CP . \cos NP = \cos CN$, and *the angle CPN is obviously a right angle.* For the sake of brevity we may call this particular curve *the sphero conic on CN.*

As ζ continues to increase, the ζ curves lie inside the sphero conic on CN and ζ is a positive maximum when $\rho = c$, that is, at the point T. The value of ζ at T is $\cos^2 n / \cos 2n$, and the curve is then reduced to the solitary point T.

In the figure the dotted lines represent the forms of the ζ curves. The points (A, B), (L, M) lie on IFJ, $IF'J$ respectively.

16. To TRACE THE CURVES u, v, w. Returning now to the conditions of stability (art. 7), the fundamental equations u, v may, by using art. 11, be written in the forms

$$u = (p_2^2 + p_3^2)\,\xi + (p_3^2 + p_1^2)\,\eta + (p_1^2 + p_2^2)\,\zeta,$$

$$v = p_2^2 p_3^2 \xi + p_3^2 p_1^2 \eta + p_1^2 p_2^2 \zeta,$$

$$w = u^2 - 4v.$$

In order that the motion may be unstable it is necessary that one at least of the expressions (uvw) should be negative (art. 5). It is evident that the curves $u = 0$, $v = 0$ cannot be real unless one at least of the coordinates $\xi\eta\zeta$ is negative at every point of the curve. The same results will presently be proved for the curve $w = 0$. *These three curves must lie within the area of the three lunes formed by producing the sides (AB, LM), (BC, MN), (CA, NL).* This is just the space

which becomes evanescent when the forces form a conservative system, and the principal axes OA, OB, OC are therefore at right angles.

17. The quartic expression

$$w = u^2 - 4v(\xi + \eta + \zeta) \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

may be written in several other forms. Substitute for u and v from art. 16, and we find

$$w = (P_1^2\xi + P_2^2\eta - P_3^2\zeta)^2 + 4P_2^2P_3^2\eta\zeta \ldots\ldots\ldots (2),$$

where $P_1^2 = p_2^2 - p_3^2$, $P_2^2 = p_1^2 - p_3^2$, $P_3^2 = p_1^2 - p_2^2$, and if p_1^2, p_2^2, p_3^2 are in descending order of magnitude, P_1^2, P_2^2, P_3^2 are positive quantities such that

$$P_1^2 - P_2^2 + P_3^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It follows that w cannot be negative unless η, ζ have opposite signs, and similarly ξ, ζ have opposite signs. Hence one at least of the three coordinates $\xi\eta\zeta$ must be negative at every point of the curve. *The spherical area enclosed by* $w = 0$ *lies therefore within the lunes formed by the sides* (AB, LM), (BC, MN), (CA, NL), and these lunes are evanescent when the principal axes OA, OB, OC are at right angles.

18. Expanding the expression (2), we find

$$w = P_1^4\xi^2 + P_2^4\eta^2 + P_3^4\zeta^2 + 2P_1^2P_2^2\xi\eta + 2P_2^2P_3^2\eta\zeta - 2P_1^2P_3^2\xi\zeta.$$

The quartic $w = 0$ *therefore touches each of the sides of the triangle* ABC *and therefore also those of* LMN. By putting ξ, ζ, η in succession equal to zero, we find that the points of contact with BC, AB, AC, &c. respectively lie on the curves

$$P_2^2\eta + P_3^2\zeta = 0, \quad P_1^2\xi + P_3^2\zeta = 0, \quad P_1^2\xi - P_3^2\zeta = 0.$$

The exact positions of these points may be deduced from the rule in art. 11 and depend on the values of P_1^2, P_2^2, P_3^2.

We notice, however, that each of these curves correspond to two points of contact such that η, ζ and ξ, ζ have opposite signs in the two first, while ξ, ζ have the same sign in the third. The two first points of contact therefore lie within the lunes, and the third is external to all three lunes.

Other properties of the curve $w = 0$ are suggested by the corresponding theorems in trilinear coordinates and need not be mentioned here.

19. A third form for w is

$$w = P_2^2 P_3^2 (1-\xi)^2 - P_1^2 P_3^2 (1-\eta)^2 + P_1^2 P_2^2 (1-\zeta)^2,$$

where the sum of the reciprocals of the coefficients is easily seen to be zero (art. 17). The quartic curve $w=0$ therefore touches each of the quadrics $P_1(1-\eta) \pm P_2(1-\zeta)=0$ in points which lie on the sphero conic described on CN (art. 15).

THE BINOMIAL THEOREM.

By A. C. Dixon, Sc.D., F.R.S.

IN the *Quarterly Journal*, Vol. XXXVIII., p. 108, Mr. Barnes has called attention to the most general case of the binomial theorem, and given a proof. The present paper gives another proof, which depends only on principles that are given in the text books of Chrystal and Hobson, and establishes the logarithmic and binomial theorems with little more trouble than is necessary for the proof of the logarithmic series only.

It may be noted that in addition to Abel's, a proof depending on the rearrangement of a double series is indicated by Chrystal (*Algebra*, Vol. 2, p. 302, ex. 19), while the theorem is a case of Taylor's and could therefore also be proved by the integration of

$$\frac{(1+z)^m}{z-t}\,dz$$

round a circular path enclosing the origin.

Let $$f(m, z) = 1 + mg(m, z),$$

where $$g(m, z) = z + \frac{m-1}{2} z^2 + \frac{(m-1)(m-2)}{2.3} z^3 + \ldots,$$

m, z being complex, and $|z|<1$. It is to be proved that

$$f(m, z) = (1+z)^m.$$

Now, if $|m|<1$, $|m-n+1|<n$ for positive integral values of m, and thus the moduli of the coefficients in $g(m, z)$ are all <1. Hence, if $|z|<b<1$, the moduli of the terms in $g(m, z)$ are less than the corresponding terms in

$$b + b^2 + b^3 + \ldots.$$

Thus, when $|z|<b$, $|m|<1$, the series $g(m, z)$ is absolutely and uniformly convergent, and the modulus of its sum $<b/(1-b)$.

If $|m|<1-b$, then the absolute value of $f(m, z)-1$, that is, of $mg(m, z)$, is less than b, and the point representing $f(m, z)$ lies within the circle $|z-1|=b$.

Suppose then that $|z|<b$, $|m|<1-b$.

Let $\log f(m, z)=K+\iota L$. Since the series $f(m, z)$ is uniformly convergent, its sum is a continuous function of z, and the logarithm $K+\iota L$ is also continuous since $f(m, z)$ does not approach the value 0, but lies within the circle $|z-1|=b$. Hence $K+\iota L$ is completely fixed when we say that it is to vanish with z.

Similarly, let $\log f(m/q, z)=k+\iota l$, where q is a positive integer, and $k+\iota l$ vanishes with z and is a continuous function of z.

We have $\{f(m/q, z)\}^q=f(m, z)$, by multiplication of series, and therefore $K+\iota L-q(k+\iota l)=$ a multiple of $2\iota\pi$, which must be 0, since it vanishes with z and is a continuous function of z. Hence $k=K/q$, $l=L/q$.

Now

$$f(m/q, z)=e^k(\cos l+\iota\sin l)$$

$$=e^{K/q}(\cos L/q+\iota\sin L/q).$$

Therefore

$$g(m/q, z)=[e^{K/q}(\cos L/q+\iota\sin L/q)-1]\,q/m$$

$$=(K+\iota L)/m$$

in the limit when q is infinite.

But, on account of the uniform convergency,

$$\operatorname*{Lim}_{q=\infty} g(m/q, z)=g(0, z);$$

that is

$$K+\iota L=mg(0, z)$$

$$=m(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots),$$

$$f(m, z)=\exp m(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots).$$

This result has been proved when $|z|<b$, $|m|<1-b$.

For a greater value of $|m|$, take a positive integer r which is $> |m|/(1-b)$. Then the result holds for m/r, that is,

$$f(m/r, z) = \exp m/r\,(z - \tfrac{1}{2}z^2 + \ldots),$$

and $$f(m, z) = \{f(m/r, z)\}^r = \exp m\,(z - \tfrac{1}{2}z^2 + \ldots).$$

The thing is therefore proved for all values of m.

In particular

$$\exp(z - \tfrac{1}{2}z^2 + \ldots) = f(1, z) = 1 + z.$$

Thus, if $|z| < 1$,

$$\log(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots,$$

and $$f(m, z) = \exp m \log(1+z),$$

the logarithm being in each case a continuous function of z and vanishing with z, so that the coefficient of ι in it is between $\pm\frac{1}{2}\pi$. These are the desired results.

When $|z| = 1$, we only need to use Abel's theorem, given in the text-books, that if a series in ascending integral powers of z is convergent when $z = 1$, it is uniformly convergent for real positive values from 0 to 1 inclusive, and that therefore its sum is a continuous function of z for such values.

If $m = \alpha + \iota\beta$, the ordinary tests show that the series $f(m, z)$ converges absolutely when $\alpha > 0$, and that the successive terms diminish without limit* when $\alpha > -1$. The binomial theorem then follows at once if $\alpha > 0$. If $\alpha > -1$, but ≤ 0, let $f_n(m, z)$ denote the sum of the first n terms in $f(m, z)$. Then, by the usual multiplication,

$$(1+z) f_n(m, z) = f_{n+1}(m+1, z) - \text{the } (n+1)^{\text{th}} \text{ term in } f(m, z)$$

$$= f(m+1, z), \text{ in the limit when } n \text{ is } \infty,$$

since $f(m+1, z)$ is convergent and the terms in $f(m, z)$ decrease without limit. It follows that $f(m, z)$ is convergent, unless $z = -1$, so that the binomial theorem holds when $|z| = 1$, provided the real part of $m+1$ is positive.

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* The terms $u_1, u_2, u_3, \ldots$ diminish without limit if $n\{|u_n/u_{n+1}| - 1\}$ tends to a positive limiting value.

Belfast,
May, 1907.

ON GENERALIZED LEGENDRE FUNCTIONS.

By E. W. Barnes, Sc.D.

§ 1. THE ordinary Legendre Functions are solutions of the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0.$$

The two fundamental solutions are $P_n(x)$ and $Q_n(x)$. When n is a positive integer, the former is a polynomial in x.

The associated functions $P_n^m(x)$ and $Q_n^m(x)$, when m is a positive integer, are defined by the equalities

$$\left.\begin{matrix}P_n^m(x)\\Q_n^m(x)\end{matrix}\right\}=(x^2-1)^{\frac{1}{2}m}\frac{d^m}{dx^m}\left\{\begin{matrix}P_n(x)\\Q_n(x)\end{matrix}\right. \quad \ldots\ldots\ldots(1).$$

They satisfy the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}y=0.$$

When m has any general value the generalised Legendre functions $P_n^m(x)$, $Q_n^m(x)$ are defined to be the solutions of this equation which reduce to their values given by (1) when m is a positive integer. By finding such solutions the generalised functions are usually obtained. But they may be considered from another point of view, in which we endeavour to generalise the idea of differentiation to the case when m is a complex quantity. Such generalisations* usually are based on the theory of contour integrals: if we can express $f(x)$ in the form $\int\phi(s)x^s\,ds$, where $\phi(s)$, and the contour of integration are independent of x, then a generalisation of $\frac{d^m}{dx^m}f(x)$, when m is complex will be given by $\int\phi(s)\frac{\Gamma(s+1)}{\Gamma(s-m+1)}x^{s-m}\,ds$, provided the latter integral has a meaning.

Now $P_n(x)$ is a particular hypergeometric function expressible in the form $F\{-n, n+1; 1; \frac{1}{2}(1-x)\}$. It therefore

* A short summary of such investigations is given in the *Encyklopädie der Mathematischen Wissenschaften*, Bd. 2, pp. 116–119.

follows by the general theory which I have developed in an earlier paper* that

$$\frac{\pi}{\sin n\pi} P_n(x) = \frac{1}{2\pi\iota} \int \frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1+s)} \{\tfrac{1}{2}(x-1)\}^s\, ds,$$

provided $|\arg(x-1)| < \pi$. This is an integral of precisely the type required, and therefore the equality

$$\frac{\pi}{\sin n\pi} P_n^m(x) = (x^2-1)^{\frac{1}{2}m} \frac{1}{2\pi\iota} \int \frac{\Gamma(s-n)\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1-m+s)\,2^s} (x-1)^{s-m}\, ds$$

may be used to define $P_n^m(x)$ for *all* values of n and m.

Similar considerations apply to $Q_n^m(x)$. We therefore arrive at contour integrals for $P_n^m(x)$ and $Q_n^m(x)$ of a very simple type. It is from such contour integrals that we develop the whole theory of generalised Legendre functions in the present paper.

Part I.

The general theory of the transformation of the series for Legendre functions.

§ 2. The transformation-theory of generalised Legendre functions is known to lead, when discussed by circuits of Pochhammer's type, to exceedingly tedious algebra, where thepossibility of mistakes is very easy. The theory given by Heine† is faulty: the only complete investigation that has been given is due to Hobson,‡ whose memoir is the standard authority. I have calculated that the methods introduced in the present paper diminish the labour of the theory discussed in Part I. of the present investigation to about one-third or one-quarter of that which previous methods have imposed.

§ 3. There are several interesting considerations involved in the transformation-theory of generalised Legendre functions. Such functions are always associated with hypergeometric series or with the equivalent Riemann P-functions of a special type. In this type of P-function the difference of the two

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* Barnes, "A new development of the theory of the hypergeometric functions." Read before the London Mathematical Society, January, 1907.

† Heine, *Handbuch der Kugelfunctionen* (1878). Hobson has discussed Heine's theory in some detail.

‡ Hobson, *Philosophical Transactions of the Royal Society* (1896), Vol. 187 (A), pp. 443–531.

exponents associated with one of the singularities is equal to one-half. Therefore the P-function is such that we have Riemann's transformation*

$$P\begin{Bmatrix} 0, & \infty, & 1 \\ 0, & \beta, & \gamma, & x \\ \frac{1}{2}, & \beta', & \gamma' \end{Bmatrix} = P\begin{Bmatrix} -1, & \infty, & 1 \\ \gamma, & 2\beta, & \gamma, & \sqrt{x} \\ \gamma', & 2\beta', & \gamma' \end{Bmatrix} \quad \ldots\ldots(A),$$

where $$\beta+\beta'+\gamma+\gamma'=\tfrac{1}{2}.$$

From this transformation Riemann deduces that we can express in terms of one another the functions

$$P(\mu, \nu, \tfrac{1}{2}, x_2), \quad P(\mu, 2\nu, \mu, x_1), \quad P(\nu, 2\mu, \nu, x_3),$$

where $$x_2 = 4x_1(1-x_1) = \frac{1}{4x_3(1-x_3)}.$$

Thus whereas the ordinary hypergeometric equation admits 24 solutions, the corresponding equation associated with the generalised Legendre functions will admit three times as many. We have in fact series whose arguments are any functions derivable by homographic transformation of the system x, $1/x$, $1-x$, $1/(1-x)$, $x/(x-1)$, $(x-1)/x$ from the functions $(1-x)/2$, $1/x^2$, $\{x-\sqrt{(x^2-1)}\}^2$. It has been customary hitherto to derive the series corresponding to such transformations from the transformed differential equation (such was Kummer's† orignal method), and then to find the actual values of the constants involved by considering some special point of the plane. This process Heine applies to series of arguments $1/x^2$ and $\{x-\sqrt{(x^2-1)}\}^2$: Hobson obtains the direct transformation of series of arguments $(1-x)/2$ and $1/x^2$. In Part I. of the present paper all the transformations are obtained directly without recourse to the differential equation, and the work is by no means laborious. And, further, the same method enables us to deduce by direct transformation the most general form of Riemann's relation (A).

Considerations of space prevent me from giving the whole of the 72 series and the expressions in terms of them of $P_n^m(x)$ and $Q_n^m(x)$. The series themselves may be found in Oldbricht's memoir,‡ though he does not express in terms of them the associated Legendre functions as they are usually defined. In Part I. of the present paper all the important expressions in series of the associated Legendre functions are

* Riemann, *Oeuvres Mathématiques* Paris (1898), pp. 73–75.

† Kummer, *Crelle*, T. 15 (1835), pp. 39–83 and 127–172.

‡ Oldbricht, *Nova Acta der Ksl. Leop-Carol. Deutschen Akademie der Naturforscher*, Bd. 52, No. 1 (Halle 1887), pp. 17–20.

given, and from the given contour integrals any particular form desired can be at once written down.

Very full references to the literature of this part of the subject are given by Oldbricht*: for the varying notations Hobson's memoir† may be consulted. After great hesitation I have decided to make one change from Hobson's notation. I replace his function $Q_n^m(x)$ by $Q_n^m(x)\frac{\sin n\pi e^{m\pi\iota}}{\sin(n+m)\pi}$. Each function has the same value when m is a positive integer, and is therefore admissible as a generalisation, and my own form arises naturally and introduces a considerable simplification into the formulæ.

Definitions of $P_n^m(x)$ *and* $Q_n^m(x)$.

§ 4. The differential equation of the associated Legendre functions is

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}y=0.$$

We assume that m and n are the most general or real complex quantities. We will first show that *the integrals*

$$I_m(x)=-\frac{1}{2\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\int\frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1-m+s)}\{\tfrac{1}{2}(x-1)\}^s ds,$$

$I_{-m}(x)$= *same expression with* $-m$ *written for* m,

$$J_m(x)=\frac{-1}{2\pi\iota}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\int\frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1-m+s)}\{-\tfrac{1}{2}(x+1)\}^s ds,$$

and $J_{-m}(x)$, *are all solutions of this equation.* In the first two integrals we assume that $|\arg(x-1)|<\pi$, and in the last two that $|\arg(-x-1)|<\pi$. The integrals have contours parallel to the imaginary axis with loops if necessary to ensure that positive sequences of poles of the subject of integration lie to the right of the contour, and negative sequences to the left. Under the assigned limitations the integrals are, by the theory developed in my previous paper, convergent.

We will consider the integral $I_m(x)$. Denoting it by I, we have, since we may differentiate with respect to x under

* *Loc. cit.*, pp. 7–11.
† *Loc. cit.*, p. 446.

the sign of integration,

$$(1-x^2)\frac{dI}{dx} = mI + \tfrac{1}{2}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}$$

$$\times \frac{(1-x^2)}{2\pi\iota}\int \frac{\Gamma(s+1-n)\,\Gamma(n+2+s)\,\Gamma(-s)}{\Gamma(2-m+s)}\{\tfrac{1}{2}(x-1)\}^s\,ds.$$

In obtaining this formula we have replaced s by $(s+1)$; this is equivalent to moving the contour of integration through a distance unity parallel to itself, as is obviously possible, since we may so choose the original contour that we pass over no poles of the subject of integration during the motion.

We therefore have

$$\frac{d}{dx}\left\{(1-x^2)\frac{dI}{dx}\right\} - \frac{m^2 I}{1-x^2}$$

$$= \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}(x-m)\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(s+1-n)\Gamma(n+2+s)\,\Gamma(-s)}{\Gamma(2-m+s)}\{\tfrac{1}{2}(x-1)\}^s ds$$

$$+ \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\tfrac{1}{4}(1-x^2)\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(s+1-n)\Gamma(n+2+s)\Gamma(1-s)}{\Gamma(2-m+s)}\{\tfrac{1}{2}(x-1)\}^{s-1} ds$$

$$= \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1-m+s)}\,\Theta\{\tfrac{1}{2}(x-1)\}^s ds,$$

where

$$\Theta = -2s + (1-m)\frac{(s-n)(s+n+1)}{1-m+s} - (1-s)(-s) + \frac{(s-n)(s+n+1)\,s}{1-m+s}$$

$$= -n(n+1).$$

Therefore I_m satisfies the differential equation. By symmetry I_{-m} satisfies the differential equation, and a similar method of proof is applicable to J_m and J_{-m}, since

$$J_m(x) = I_m(-x).$$

§ 5. *We now define the first associated Legendre function* $P_n^m(x)$ *for general complex values of* m *and* n *by the equality**

$$P_n^m(x) = -\frac{\sin n\pi}{\pi} I_m(x).$$

Our definition is thus valid when $|\arg(x-1)| < \pi$, that is to say, for the whole plane of the variable x dissected by a cross cut along the real axis from $-\infty$ to $+1$. This cross

* The case when n is an integer is discussed in § 13

cut serves to define the function $\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}$ which occurs in the definition of $I_m(x)$. We assume that $\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} = \frac{(x+1)^{\frac{1}{2}m}}{(x-1)^{\frac{1}{2}m}}$, and each of the functions in this quotient is uniquely defined, provided $|\arg(x+1)|<\pi$ and $|\arg(x-1)|<\pi$.

In the first place we see that immediately from the definition we have

$$P_n^m(x) = P^m_{-n-1}(x).$$

In the second place we see that, when $m=0$,

$$P_n(x) = \frac{1}{2\pi\iota}\,\frac{\sin n\pi}{\pi}\int \frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(1+s)}\{\tfrac{1}{2}(x-1)\}^s ds,$$

and therefore, if m be a positive integer,

$$\frac{d^m P_n(x)}{dx^m} = \frac{1}{2\pi\iota}\,\frac{\sin n\pi}{\pi}\int \frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(s-m+1)\,2^s}(x-1)^{s-m} ds$$

$$= (x^2-1)^{-\frac{1}{2}m} P_n^m(x).$$

§ 6. *Expansions of* $P_n^m(x)$ *in powers of* $(1\pm x)/2$ *and* $2/(1\pm x)$.

Suppose that $|1-x|<2$. Then we may bend round the contour of the integral I_m so as to include the positive sequence of poles of the subject of integration, and we have, by Cauchy's theorem,

$$P_n^m(x) = \frac{1}{\Gamma(1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} F\{-n, n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\},$$

which is the expansion of $P_n^m(x)$ in powers of $\frac{1}{2}(1-x)$ when $|1-x|<2$. This is Hobson's definition [(5), p. 451].

Suppose next that $|1-x|>2$. Then we may bend round the contour of the integral I_m so as to include the two negative sequences of poles of the subject of integration, and we find

$$P_n^m(x) = \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\left[\frac{\Gamma(2n+1)}{\Gamma(1+n)\Gamma(1-m+n)}\{\tfrac{1}{2}(x-1)\}^n F\left\{-n, m-n; -2n; \frac{2}{1-x}\right\}\right.$$

$$\left. + \frac{\Gamma(-2n-1)}{\Gamma(-m-n)\Gamma(-n)}\{\tfrac{1}{2}(x-1)\}^{-n-1} F\left\{n+1,\ 1+m+n;\ 2+2n;\ \frac{2}{1-x}\right\}\right],$$

the expansion of $P_n^m(x)$ in powers of $\frac{2}{1-x}$ when $|1-x|>2$.

Again (by § 16 H. F.)*

$$\int\Gamma(-s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,(1-\xi)^s ds$$

$$=\Gamma(\rho-\alpha_1)\,\Gamma(\rho-\alpha_2)\int\frac{\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(-s)}{\Gamma(\rho+s)}(-\xi)^s ds.$$

Hence

$$P_n^m(x)=\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\frac{\sin n\pi}{\pi\,\Gamma(1-m+n)\,\Gamma(-m-n)}$$

$$\times\frac{1}{2\pi\iota}\int\Gamma(-s)\,\Gamma(-m-s)\,\Gamma(-n+s)\,\Gamma(n+1+s)\left\{\tfrac{1}{2}(1+x)\right\}^s ds \qquad \ldots\ldots(1).$$

This integral yields immediately expansions for $P_n^m(x)$ in ascending or descending powers of $\{\frac{1}{2}(1+x)\}$ as $|1+x|\lesssim 2$.

We get

$$P_n^m(x)=\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\frac{\Gamma(-m)}{\Gamma(1-m+n)\,\Gamma(-m-n)}F\{-n,\,n+1;\,1+m;\,\tfrac{1}{2}(1+x)\}$$

$$+\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\frac{e^{\mp m\pi\iota}\,\Gamma(m)}{\Gamma(-n)\,\Gamma(1+n)}F\{-n,\,n+1;\;1-m;\;\tfrac{1}{2}(1+x)\},$$

and analogous series, the upper or lower sign being taken as $I(x)$ is positive or negative.

§ 7. *Expansions of* $P_n^m(x)$ *in powers of* $\dfrac{x-1}{x+1}$ *and* $\dfrac{x+1}{x-1}$.

From the equality (1) of § 6 we obtain at once

$$P_n^m(x)=-\frac{1}{2\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\left\{\tfrac{1}{2}(1+x)\right\}^n$$

$$\times\frac{1}{\Gamma(-n)\,\Gamma(-m-n)}\int\frac{\Gamma(-s)\,\Gamma(s-n)\,\Gamma(s-m-n)}{\Gamma(1-m+s)}\left(\frac{1-x}{1+x}\right)^s ds,$$

provided $\left|\arg\dfrac{1-x}{1+x}\right|<\pi$.

Therefore

$$P_n^m(x)=\frac{1}{\Gamma(1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\left\{\tfrac{1}{2}(1+x)\right\}^n F\left\{-n,\;-m-n;\;1-m;\;\frac{x-1}{x+1}\right\}$$

$$=\frac{1}{\Gamma(1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\left\{\tfrac{1}{2}(1+x)\right\}^{-n-1}F\left\{n+1,\;1-m+n;\;1-m;\;\frac{x-1}{x+1}\right\},$$

* In this way reference will be made to the author's previous paper quoted in § 1. Hobson's paper will be referred to simply by the letter H.

when $$\left|\frac{x-1}{x+1}\right|<1.$$

By taking the integral to enclose the negative sequences of poles we get the two terms of the analogous expansions valid when

$$\left|\frac{x-1}{x+1}\right|>1.$$

§ 8. *We now define the second associated Legendre function* $Q_n^{m}(x)$ *for the general complex values of* m *and* n *by the equality*

$$2\,Q_n^{m}(x)=J_m-e^{\mp n\pi\iota}I_m,$$

the upper or lower sign being taken as $I(x)$, the imaginary part of x, is positive or negative, I_m and J_m being the integrals introduced in § 4.

The definition seems at this stage to be both arbitrary and artificial. We shall see, however, in § 11 that $Q_n^{m}(x)$, as thus defined, is expressible either by a simple contour integral or by a single hypergeometric series. And in § 21 we shall express $Q_n^{m}(x)$ as a hypergeometric series of argument $1/x^2$, which reduces when $m=0$, to the series by which $Q_n(x)$ is usually defined in the elementary theory of Legendre functions.

When $m=0$ we have

$$Q_n(x)=\frac{1}{2\pi\iota}\int\frac{\Gamma(s-n)\Gamma(n+1+s)\Gamma(-s)}{\Gamma(1+s)\,2^{s+1}}\{(x-1)^s e^{\mp n\pi\iota}-(-x-1)^s\}ds.$$

Hence, if m be a positive integer,

$$(x^2-1)^{\frac{1}{2}m}\frac{d^m Q_m(x)}{dx^m}=\frac{1}{2\pi\iota}e^{\mp n\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\int\frac{\Gamma(s-n)\Gamma(n+1+s)\Gamma(-s)}{\Gamma(s+1-m)2^{s+1}}(x-1)^s ds$$

$$-\frac{1}{2\pi\iota}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\int\frac{\Gamma(s-n)\,\Gamma(n+1+s)\,\Gamma(-s)}{\Gamma(s+1-m)\,2^{s+1}}(-x-1)^s ds$$

$$=Q_n^{m}(x).$$

From the definition we have, when $|1\pm x|<2$, the expansion [H. (15), p. 460]

$$2\,Q_n^{m}(x)\frac{\Gamma(1-m)\sin n\pi}{\pi}=e^{\mp n\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}F\{-n,n+1;\,1-m;\,\tfrac{1}{2}(1-x)\}$$

$$-\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}F\{-n,n+1;\,1-m;\,\tfrac{1}{2}(1+x)\}.$$

§ 9. *The expansion of* $Q_n^m(x)$ *when* $|(1-x)|<2$.

Again, by H. F. (§ 12), we have the formula, already substantially obtained in the transformations for $P_n^m(x)$,

$$\frac{\Gamma(-n)\,\Gamma(1+n)}{\Gamma(1-m)}F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1+x)\}$$

$$=\frac{\Gamma(-m)\,\Gamma(-n)\,\Gamma(1+n)}{\Gamma(1-m+n)\,\Gamma(-m-n)}F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\}$$

$$+\Gamma(m)\left(\frac{1+x}{1-x}\right)^m F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\}.$$

Therefore

$$J_m=\frac{\Gamma(-m)\,\Gamma(1+m)}{\Gamma(1-m+n)\,\Gamma(-m-n)}I_{-m}+\frac{\Gamma(m)\,\Gamma(1-m)}{\Gamma(-n)\,\Gamma(1+n)}I_m e^{\pm m\pi\iota},$$

for $(x-1)^m=(1-x)^m e^{\pm m\pi\iota}$, the upper or lower sign being taken as $I(x)$ is positive or negative.

Hence

$$2Q_n^m(x)\sin m\pi=-\{\sin n\pi e^{\pm m\pi\iota}+\sin m\pi e^{\mp n\pi\iota}\}I_m$$

$$-\frac{\pi}{\Gamma(1-m+n)\,\Gamma(-m-n)}I_{-m}$$

$$=-\sin(n+m)\,\pi I_m-\frac{\pi}{\Gamma(1-m+n)\,\Gamma(-m-n)}I_{-m}.$$

Thus

$$\frac{2Q_n^m(x)\sin m\pi\Gamma(-m-n)}{\pi}=\frac{I_m}{\Gamma(1+n+m)}-\frac{I_{-m}}{\Gamma(1-m+n)} \qquad \ldots\ldots(1).$$

Hence, when $|1-x|<2$, we have the expansion

$$-2Q_n^m(x)\frac{\sin n\pi}{\pi}\Gamma(-m-n)$$

$$=\frac{\Gamma(m)}{\Gamma(1+m+n)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\}$$

$$+\frac{\Gamma(-m)}{\Gamma(1+n-m)}\left(\frac{x+1}{x-1}\right)^{-\frac{1}{2}m}F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\}.$$

From the relation (1) we readily see that

$$Q_n^m(x)\,\Gamma(-m-n)=Q_n^{-m}(x)\,\Gamma(m-n).$$

§ 10. *Expansion of* $Q_n^m(x)$ *when* $|1+x|<2$.

We can easily deduce from our previous results that

$$I_m = \frac{\Gamma(-m)\,\Gamma(1+m)}{\Gamma(1-m+n)\,\Gamma(-n-m)}\,J_{-m} + \frac{\Gamma(m)\,\Gamma(1-m)}{\Gamma(-n)\,\Gamma(n+1)}\,J_m e^{\mp m\pi\iota}.$$

We therefore have

$$2\,Q_n^m(x)\sin m\pi = J_m\{\sin m\pi + \sin n\pi e^{\mp(m+n)\pi\iota}\} + \frac{\pi e^{\mp n\pi\iota}\,J_{-m}}{\Gamma(1-m+n)\,\Gamma(-n-m)}.$$

We therefore have

$$2\,Q_n^m(x)\,e^{\pm n\pi\iota}\,\Gamma(-m-n)\,\frac{\sin m\pi}{\pi} = \frac{J_{-m}}{\Gamma(1-m+n)} - \frac{J_m}{\Gamma(1+m+n)}.$$

Hence, when $|1+x|<2$,

$$2\,Q_n^m(x)\,\frac{\sin n\pi}{\pi}\,\Gamma(-m-n)\,e^{\pm n\pi\iota}$$

$$= \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\frac{\Gamma(-m)}{\Gamma(1-m+n)}\,F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1+x)\}$$

$$+ \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\frac{\Gamma(m)}{\Gamma(1+m+n)}\,F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1+x)\}.$$

From this result, coupled with that of the previous paragraph, we obtain

$$Q_n^m(-x) = -\,Q_n^m(x)\,e^{\pm n\pi\iota} \quad [\text{H. (21), p. 463}].$$

Again, by § 6,

$$P_n^m(-x) = \frac{1}{\Gamma(1-m)}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1+x)\}$$

$$= \frac{1}{\Gamma(1-m)}e^{\mp n\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\} - 2\,Q_n^m(x)\frac{\sin n\pi}{\pi}$$

by § 8.

Therefore

$$P_n^m(-x) = P_n^m(x)\,e^{\mp n\pi\iota} - 2\,Q_n^m(x)\,\frac{\sin n\pi}{\pi} \quad [\text{H. 20, p. 463}].$$

§ 11. *We will now show that*

$$2\,Q_n^m(x)\frac{\sin n\pi}{\sin(m+n)\pi}$$

$$= -\,\frac{1}{2\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\int\frac{\Gamma(s+n+1)\,\Gamma(m-s)\,\Gamma(-s)}{\Gamma(n+1-s)}\,\{\tfrac{1}{2}(x-1)\}^s\,ds,$$

provided $|\arg(x-1)|<\pi$, *the contour being taken parallel to the imaginary axis with loops to ensure that positive sequences lie to the right and negative sequences to the left of the contour.*

The theorem is important as it enables us to express $Q_n^m(x)$ as a single contour integral of the same type as the integral for $P_n^m(x)$ given in § 6.

Suppose that $|\frac{1}{2}(x-1)|<1$. Then we may bend round the contour to the right and we see that the previous integral is equal to

$$\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\Big[\Gamma(m)\,F\{n+1,-n;\ 1-m;\ \tfrac{1}{2}(1-x)\}$$

$$+\left(\frac{x-1}{2}\right)^{m}\frac{\Gamma(m+n+1)\Gamma(-m)}{\Gamma(n+1-m)}F\{n+m+1,\ m-n;\ 1+m;\ \tfrac{1}{2}(1-x)\}\Big]$$

$$=\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\Gamma(m)\,F\{n+1,-n;\ 1-m;\ \tfrac{1}{2}(1-x)\}$$

$$+\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\frac{\Gamma(m+n+1)\,\Gamma(-m)}{\Gamma(n+1-m)}F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\}$$

$$=2\,Q_n^m(x)\,\frac{\sin n\pi}{\sin(m+n)\,\pi},\ \text{by § 9.}$$

We can now at once deduce hypergeometric series for $Q_n^m(x)$ *valid either when* $|1-x|>2$, *or when* $|1+x|>2$, *and in each case* $Q_n^m(x)$ *is expressed by a single hypergeometric series.*

When $|1-x|>2$ we may bend the contour of the integral just considered round to the left, and we get

$$2\,Q_n^m(x)\,\frac{\sin n\pi}{\sin(m+n)\,\pi}$$

$$=\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\frac{\Gamma(n+1)\,\Gamma(m+n+1)}{\Gamma(2n+2)}\{\tfrac{1}{2}(x-1)\}^{-n-1}$$

$$\times F\left\{m+n+1,\ n+1;\ 2n+2;\ \frac{2}{1-x}\right\}.$$

Thus

$$Q_n^m(x)=\cot n\pi\,\frac{\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}-n)}{2^{2n+2}\,\Gamma(-m-n)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\left(\frac{2}{x-1}\right)^{n+1}$$

$$\times F\left\{m+n+1,\ n+1;\ 2n+2;\ \frac{2}{1-x}\right\}$$

$$=\cot n\pi\,\frac{\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}-n)}{2^{2n+2}\,\Gamma(-m-n)}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\left(\frac{2}{x-1}\right)^{n+1}$$

$$\times F\left\{n+1,\ n+1-m;\ 2n+2;\ \frac{2}{1-x}\right\}.$$

And therefore [H. F. § 11 (A)] when $|1+x|>2$,

$$Q_n^m(x) = \cot n\pi \frac{\sqrt{\pi}\,\Gamma(-\frac{1}{2}-n)}{2^{2n+2}\Gamma(-m-n)} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \left(\frac{2}{x+1}\right)^{n+1}$$

$$\times F\left\{n+1,\ n+m+1;\ 2n+2;\ \frac{2}{1+x}\right\} \ldots\ldots (1)$$

$$= \cot n\pi \frac{\sqrt{\pi}\,\Gamma(-\frac{1}{2}-n)}{2^{2n+2}\Gamma(-m-n)} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \left(\frac{2}{x+1}\right)^{n+1}$$

$$\times F\left\{n+1,\ n-m+1;\ 2n+2;\ \frac{2}{1+x}\right\}.$$

None of these simple series appear in Hobson's memoir. Their existence might have been suspected *a priori* as, when m and n are general complex numbers, $P_n^m(x)$ and $Q_n^m(x)$ are functions of the same degree of complexity each with a simple expression in a suitable region of the x plane. This will be more apparent in the sequel when the Kummer-Riemann transformations are considered.

From the expression (1) we see that, provided

$$\left|\arg\left(\frac{-2}{1+x}\right)\right| < \pi,$$

$$2\,Q_n^m(x)\frac{\sin n\pi}{\sin(m+n)\,\pi}$$

$$= -\frac{1}{2\pi\iota}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\left(\frac{2}{1+x}\right)^{n+1}\int \frac{\Gamma(s+n+1)\Gamma(s+n+m+1)\Gamma(-s)}{\Gamma(2n+2+s)}\left(\frac{-2}{1+x}\right)^{s} ds$$

$$= \frac{1}{2\pi\iota}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} e^{\mp\pi\iota n}\int \frac{\Gamma(s)\,\Gamma(s+m)\,\Gamma(n+1-s)}{\Gamma(n+1+s)}\left(\frac{-2}{1+x}\right)^{s} ds.$$

If we bend round the integral to the left we get, when $|1+x|<2$,

$$-2\,Q_n^m(x)\frac{\sin n\pi}{\sin(m+n)\,\pi}\,e^{\pm\pi\iota n}$$

$$= \Gamma(m)\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1+x)\}$$

$$+ \frac{\Gamma(-m)\Gamma(n+1+m)}{\Gamma(n+1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1+x)\}.$$

We thus verify the result of § 10.

§ 12. *We can now obtain the important formula*

$$Q_n^m(x) - Q^m{}_{-n-1}(x) = \pi \cot n\pi P_n^m(x) \quad [\text{H. 18, p. 462}] \ldots (1).$$

By equation (1), § 9, we have

$$2\,Q_n^m(x)\frac{\sin m\pi}{\pi} = \frac{I_m}{\Gamma(-m-n)\Gamma(1+n+m)} - \frac{I_{-m}}{\Gamma(-m-n)\Gamma(1+n-m)}.$$

When n is changed into $-n-1$, the integrals I_m and I_{-m} are unaltered, and therefore the second term on the right-hand side is unchanged.

Hence

$$2\{Q_n^m(x) - Q^m{}_{-n-1}(x)\}\frac{\sin m\pi}{\pi} = -\frac{1}{\pi}\{\sin\pi(m+n) + \sin\pi(m-n)\}I_m,$$

or $$Q_n^m(x) - Q^m{}_{-n-1}(x) = -\cos n\pi I_m = \pi \cot n\pi P_n^m(x).$$

We therefore have the given formula.

Similarly, since

$$P_n^m(x) = -\frac{\sin n\pi}{\pi} I_m(x),$$

and

$$2\,Q_n^m(x)\,\Gamma(-n-m)\frac{\sin m\pi}{\pi} = \frac{I_m}{\Gamma(1+n+m)} - \frac{I_{-m}}{\Gamma(1-m+n)},$$

we have the important identity [H. 19, p. 462]

$$2\,Q_n^m(x)\,\Gamma(-m-n)\frac{\sin m\pi \sin n\pi}{\pi^2} = \frac{P_n^{-m}(x)}{\Gamma(1-m+n)} - \frac{P_n^m(x)}{\Gamma(1+m+n)} \quad \ldots\ldots(2).$$

Formulæ (1) and (2) coupled with previous results enable us to express any of the eight solutions

$$P_n^{\pm m}(x), \quad P_{-n-1}^{\pm m}(x), \quad Q_n^{\pm m}(x), \quad Q_{-n-1}^{\pm m}(x)$$

in terms of a fundamental pair.

The formula (2) coupled with the expansions for $P_n^m(x)$ obtained in §§ 6 and 7 enables us to find expansions for $Q_n^m(x)$ when $|1 \pm x| \leqq 2$, or when $|1+x| < |1-x|$. We can thus express any of the 24 solutions of the fundamental differential equation which have arguments derived by homographic transformation from $\frac{1}{2}(1-x)$ in terms of $P_n^m(x)$ and $Q_n^m(x)$.

From the previous results we can deduce a formula which is of importance when we come to consider the asymptotic expansion of $P_n^m(x)$ when $|m|$ is large.

We will show that

$$\frac{P_n^m(x)\,\pi}{\Gamma(m-n)\,\Gamma(1+m+n)} = -P_n^{-m}(x)\,e^{\mp\pi\iota m}\sin\pi n + P_n^{-m}(-x)\sin\pi m,$$

the upper or lower sign being taken as $I(x)$ is positive or negative.

By § 10 $\quad P_n^{-m}(-x) = P_n^{-m}(x)e^{\mp n\pi\iota} - 2\,Q_n^{-m}(x)\dfrac{\sin n\pi}{\pi}$.

Therefore by § 9

$$P_n^{-m}(-x) = P_n^{-m}(x)\,e^{\mp n\pi\iota} - 2\,Q_n^{m}(x)\,\frac{\Gamma(-m-n)}{\Gamma(m-n)}\,\frac{\sin n\pi}{\pi}.$$

Hence, by the equality (2) just proved, we have

$$\frac{P_n^{-m}(x)}{\Gamma(1-m+n)} = \frac{P_n^{m}(x)}{\Gamma(1+m+n)} + \frac{\sin m\pi}{\pi}\,\Gamma(m-n)\,[P_n^{-m}(x)\,e^{\mp n\pi\iota} - P_n^{-m}(-x)].$$

Thus

$$\frac{P_n^m(x)\,\pi}{\Gamma(1+m+n)} = P_n^{-m}(x)\,\Gamma(m-n)\{\sin\pi(m-n) - \sin m\pi e^{\mp n\pi\iota}\} + \sin m\pi\,\Gamma(m-n)\,P_n^{-m}(-x)$$

$$= \sin m\pi\,\Gamma(m-n)\,P_n^{-m}(-x) - e^{\mp\pi\iota m}\sin\pi n\,\Gamma(m-n)\,P_n^{-m}(x),$$

we thus have the given result.

§ 13. *Descriptive properties of $P_n^m(x)$. Exceptional cases.* From the definition

$$P_n^m(x) = \frac{\sin n\pi}{\pi}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\frac{1}{2\pi\iota}\int\frac{\Gamma(s-n)\Gamma(s+n+1)\Gamma(-s)}{\Gamma(1-m+s)}\{\tfrac{1}{2}(x-1)\}^s ds \qquad \ldots\ldots(1),$$

we have seen that in general $P_n^m(x)$ has a cross cut from $-\infty$ to $+1$. The equivalent formula

$$P_n^m(x) = \frac{1}{\Gamma(1-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\} \qquad \ldots\ldots(2)$$

shows that the singularity of $P_n^m(x)$ at $x=1$ is of the type

$$(x-1)^{-\frac{1}{2}m}\{C_0 + C_1 x + \ldots\}.$$

The formula (2) *is a valid definition of* $P_n^m(x)$ *in all cases except when* m *is a positive integer, or when* x *lies on the cross cut* $(-\infty, 1)$.

When $m=0$, or when $m/2$ is an integer, we see that $P_n(x)$ has no singularity at $x=1$; it has a cross cut from $-\infty$ to -1.

The nature of the singularity of $P_n^m(x)$ at $x=-1$ is (§ 6) given by

$$P_n^m(x)\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}$$

$$=\frac{\Gamma(-m)}{\Gamma(1-m+n)\,\Gamma(-m-n)}F\{-n, n+1;\ 1+m;\ \tfrac{1}{2}(1+x)\}$$

$$+\{\tfrac{1}{2}(1+x)\}^{-m}\frac{\Gamma(m)}{\Gamma(-n)\,\Gamma(1+n)}F\{1-m+n, -m-n;\ 1-m;\ \tfrac{1}{2}(1+x)\}.$$

When $m=0$ we have (§ 6), if n be not a positive or negative integer or zero,

$$\left(\frac{\pi}{\sin n\pi}\right)^2 P_n(x)=-\frac{1}{2\pi\iota}\int\{\Gamma(-s)\}^2\,\Gamma(-n+s)\,\Gamma(n+1+s)\,\{\tfrac{1}{2}(1+x\ \}^s\,ds$$

$$=-\sum_{t=0}^{\infty}\frac{\Gamma(t-n)\,\Gamma(n+1+t)}{(t!)^2}\{\tfrac{1}{2}(1+x)\}^t\{\log\tfrac{1}{2}(1+x)-2\psi(1+t)+\psi(t-n)$$
$$+\psi(n+1+t)\},$$

and therefore $P_n(x)$ has a logarithmic singularity at $x=-1$ typified by

$$\frac{\sin n\pi}{\pi}F\{-n, n+1;\ 1;\ \tfrac{1}{2}(1+x)\}\log\{\tfrac{1}{2}(1+x)\}$$
$$+\text{ a power series in }\tfrac{1}{2}(1+x).$$

When n is a positive or negative integer we see from (2) that $P_n^m(x)$ is a polynomial multiplied by $\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}$.

When m is a positive integer we see from (1) that

$$P_n^m(x)=(x^2-1)^{\frac{1}{2}m}\frac{\Gamma(m+n+1)}{2^m\Gamma(1+n-m)\,\Gamma(m+1)}$$
$$\times F\{m-n, m+n+1;\ m+1;\ \tfrac{1}{2}(1-x)\}\ldots\ldots(3).$$

The latter form is valid if n be not an integer (positive, negative, or zero). When m is a positive integer and n is an integer there are three cases:—

(1) n positive and $\geq m$; the expansion (3) is valid and $P_n^m(x)\,(x^2-1)^{-\frac{1}{2}m}$ is a polynomial.

(2) n positive and $< m$; the expression (2) shows that $P_n^m(x)$ is zero; a solution of the differential equation is $\Gamma(1+n-m)\,P_n^m(x)$.

(3) n negative. In this case formula (3) may be written

$$P_n^m(x) = (x^2-1)^{\frac{1}{2}m}\frac{\Gamma(m-n)}{2^m\Gamma(-m-n)\,\Gamma(m+1)}$$

$$\times F\{m-n,\ n+m+1;\ m+1;\ \tfrac{1}{2}(x-1)\}\ldots(4).$$

This is valid if n is negative and $-n > m$. If $-n \leqslant m$ the formula shows that $P_n^m(x)$ is zero; a solution is

$$\Gamma(-m-n)\,P_n^m(x).$$

Definition of $P_n^m(x)$, when $-1 < x < 1$.

When $m=0$, we have seen that $P_n(x)$ has a cross cut from $-\infty$ to -1; when, however, $m/2$ is not an integer $P_n^m(x)$ has a cross cut from $-\infty$ to $+1$, and is therefore not defined when $-1 < x < 1$.

It is convenient to take such a definition in this case that $P_n^m(x)$ is real when m and n are real [H. 470].

We therefore assume that, in the limit when $\epsilon = 0$ and $-1 < x < 1$,

$$P_n^m(x) = e^{\frac{1}{2}m\pi\iota}P_n^m(x+\epsilon\iota) = e^{-\frac{1}{2}m\pi\iota}P_n^m(x-\epsilon\iota)$$

$$= \frac{1}{\Gamma(1-m)}\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m}F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\}.$$

Hence, when m is a positive integer and $-1 < x < 1$,

$$P_n^m(x) = (-)^m\frac{(1-x^2)^{\frac{1}{2}m}}{2^m}\,\frac{\Gamma(m+n+1)}{\Gamma(m+1)\,\Gamma(1+n-m)}$$

$$\times F\{m-n,\ m+n+1;\ m+1;\ \tfrac{1}{2}(1-x)\}.$$

We can now show that if m be zero or a positive integer, for general values of x not on the cross cut $-\infty$ to $+1$,

$$P_n^{-m}(x) = P_n^m(x)\,\frac{\Gamma(1-m+n)}{\Gamma(1+m+n)},$$

and, if $-1 < x < 1$,

$$P_n^{-m}(x) = (-)^m P_n^m(x)\,\frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

These formulæ are valid for all values of n: of course when n is a positive integer $< m$ the right-hand side becomes indeterminate.

For general values of x not on the cross cut $-\infty$ to 1, if m be zero or a positive integer, we have

$$P_n^m(x) = (x^2-1)^{\frac{1}{2}m} \frac{\Gamma(m+n+1)}{2^m \Gamma(1+n-m)\Gamma(m+1)} \times F\{m-n, m+n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\},$$

and

$$P_n^{-m}(x) = \frac{1}{\Gamma(1+m)} \left(\frac{x+1}{x-1}\right)^{-\frac{1}{2}m} F\{-n, n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\}.$$

Hence by the Kummer-Gauss relation [H. F. § 11]

$$P_n^{-m}(x) = \frac{1}{\Gamma(1+m)} \frac{(x^2-1)^{\frac{1}{2}m}}{2^m} F\{m-n, m+n+1;\ m+1;\ \tfrac{1}{2}(1-x)\}$$

$$= \frac{\Gamma(1+n-m)}{\Gamma(1+n+m)} P_n^m(x).$$

When $-1 < x < 1$, we have

$$P_n^m(x) = (-)^m (1-x^2)^{\frac{1}{2}m} \frac{\Gamma(m+n+1)}{2^m \Gamma(1+n-m)\Gamma(m+1)} \times F\{m-n, m+n+1;\ m+1;\ \tfrac{1}{2}(1-x)\},$$

and

$$P_n^{-m}(x) = \frac{1}{\Gamma(1+m)} \left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}m} F\{-n, n+1;\ m+1;\ \tfrac{1}{2}(1-x).$$

Therefore in like manner

$$P_n^{-m}(x) = (-)^m P_n^m(x) \frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

§ 14. *The function $Q_n^m(x)$ and its definition when $-1 < x < 1$.* $Q_n^m(x)$ has been defined (§ 8) by the formula

$$Q_n^m(x) = \frac{1}{2\pi\iota} e^{\mp n\pi\iota} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \int \frac{\Gamma(s-n)\Gamma(n+1+s)\Gamma(-s)}{\Gamma(s+1-m)\,2^{s+1}} (x-1)^s\, ds$$

$$- \frac{1}{2\pi\iota} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \int \frac{\Gamma(s-n)\Gamma(n+1+s)\Gamma(-s)}{\Gamma(s+1-m)\,2^{s+1}} (-x-1)^s\, ds.$$

This definition is valid for all values of m and n if x do not lie on the real axis, and n is not an integer or zero.

If n be a positive or negative integer or zero the contours of the integrals just considered cannot be drawn, but the final formula of § 8 shows that in this case $Q_n^m(x)$ is infinite, if m be not zero or a positive integer. If, however, m be zero or a positive integer the formula of § 11 may be written

$$2\,Q_n^m(x) = (-)^m \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \frac{\Gamma(n+1)\,\Gamma(m+n+1)}{\Gamma(2n+2)} \{\tfrac{1}{2}(x-1)\}^{-n-1} \times F\left\{m+n+1,\ n+1;\ 2n+2;\ \frac{2}{1-x}\right\}.$$

It therefore gives a definite value for $Q_n^m(x)$ when n is not a negative integer: it shows that $Q_n^m(x)$ is infinite when n is a negative integer and $m+n+1 \leq 0$, and further that $Q_n^m(x)$ has a definite value when n is a negative integer and $m+n+1>0$.

The final expression of § 9 shows that, in general, $Q_n^m(x)$ has a cross cut from $-\infty$ to $+1$.

It is convenient to give a definition of $Q_n^m(x)$ valid when $-1<x<1$. We take its value to be the mean of the values on either side of the cross cut, so that $Q_n^m(x)$ is real when m and n are real. We thus have, in the limit when $\epsilon = 0$,

$$Q_n^m(x) = \tfrac{1}{2}\{Q_n^m(x+\epsilon\iota) + Q_n^m(x-\epsilon\iota)\}.$$

Now by § 9 in the limit when ϵ vanishes

$$-\frac{2\sin n\pi}{\pi}\Gamma(-m-n)\,Q_n^m(x \pm \epsilon\iota)$$

$$= e^{\mp\pi\iota(\frac{1}{2}m)} \frac{\Gamma(m)}{\Gamma(1+m+n)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m} F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\}$$

$$+ e^{\pm\pi\iota(\frac{1}{2}m)} \frac{\Gamma(-m)}{\Gamma(1+n-m)} \left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}m} F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\} \quad \ldots\ldots(1).$$

Therefore, when $-1<x<1$,

$$Q_n^m(x) = -\frac{\pi\cos\frac{1}{2}m\pi}{2\sin n\pi\,\Gamma(-m-n)} \left[\frac{\Gamma(m)}{\Gamma(1+m+n)} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m} \times F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1-x)\}\right.$$

$$\left. + \frac{\Gamma(-m)}{\Gamma(1+n-m)} \left(\frac{1+x}{1-x}\right)^{-\frac{1}{2}m} F\{-n,\ n+1;\ 1+m;\ \tfrac{1}{2}(1-x)\}\right].$$

This definition differs from that of Hobson (p. 471). Each reduces to Heine's definition when $m = 0$.

By equation (1) we see that when $-1 < x < 1$ [H. 30, p. 471]

$$Q_n^m(x+\epsilon\iota)\, e^{-\pi\iota(\frac{1}{2}m)} - Q_n^m(x-\epsilon\iota)\, e^{\pi\iota(\frac{1}{2}m)} = -\frac{\iota\pi \sin(m+n)\pi}{\sin n\pi} P_n^m(x),$$

where $P_n^m(x)$ has the value defined in § 13.

The formula of § 11 gives us, when $-1 < x < 1$,

$$2\, Q_n^m(x) \frac{\sin n\pi}{\sin(m+n)\pi}$$

$$= -\frac{1}{2\pi\iota} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m} \cos\tfrac{1}{2} m\pi \int \frac{\Gamma(s+n+1)\,\Gamma(m-s)\,\Gamma(-s)}{\Gamma(n+1-s)} \{\tfrac{1}{2}(x-1)\}^s ds,$$

the contour embracing the positive sequences of poles.

For general complex values of x, the formula

$$2\, Q_n^m(x) \frac{\sin n\pi}{\sin(m+n)\pi} = \frac{-1}{2\pi\iota} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \times \int \frac{\Gamma(s+n+1)\,\Gamma(m-s)\,\Gamma(-s)}{\Gamma(n+1-s)} \{\tfrac{1}{2}(x-1)\}^s ds$$

is valid for all values of m when n is not an integer. In particular, when n is complex and m is a positive integer, $Q_n^{m-n}(x) = 0$. When m is an integer (positive, negative, or zero) the formula

$$2\, Q_n^m(x) = (-)^{m-1} \frac{1}{2\pi\iota} \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \int \frac{\Gamma(s+n+1)\,\Gamma(m-s)\,\Gamma(-s)}{\Gamma(n+1-s)} \{\tfrac{1}{2}(x-1)\}^s ds$$

is valid for all values of n, except integral values.

Expansions of $P_n^m(x)$, $Q_n^m(x)$ in series of arguments derived by homographic transformation from x^2.

§ 15. We have previously connected $P_n^m(x)$, $Q_n^m(x)$ with the 24 series derived by homographic transformation from $\frac{1}{2}(1-x)$ We will now show that we may equally connect them with 24 series derived by homographic transformation from x^2, and in this way establish one of the two Riemann transformations indicated in § 3.

At the outset we will take one of the simplest forms of such transformation, and show that, if $\alpha + \alpha' + \beta + \beta' = \frac{1}{2}$,

$$F\{\alpha+\beta,\ \alpha+\beta';\ 1-\alpha'+\alpha;\ 1-y^2\} = F\{2\alpha+2\beta,\ 2\alpha+2\beta';\ 1-\alpha'+\alpha;\ \tfrac{1}{2}(1-y)\}.$$

The limitations under which this formula is valid must be carefully noticed. The second series is convergent when $|1-y|<2$, that is to say within a circle of radius 2 surrounding the point $y=1$; and if the hypergeometric series be expressed as a multiple of the contour integral

$$-\frac{1}{2\pi\iota}\int\frac{\Gamma(2\alpha+2\beta+s)\,\Gamma(2\alpha+2\beta'+s)\,\Gamma(-s)}{\Gamma(1-\alpha'+\alpha+s)}\,\tfrac{1}{2}\{(y-1)\}^s\,ds$$

we see that it represents a function with a cross cut given by $|\arg(y-1)|=\pi$, that is a cross cut from $-\infty$ to -1. The first series, on the other hand, is convergent when $|1-y^2|<1$, or, putting $y=re^{i\theta}$, when y lies within the lemniscate $r^2=2\cos 2\theta$. Now this lemniscate consists of two ovals enclosing respectively the points ± 1, with a single point of junction at the origin. In each of these ovals the hypergeometric series may, and in point of fact does, represent a different function. And it is the function $F_1\{\alpha+\beta,\ \alpha+\beta';\ 1-\alpha'+\alpha;\ 1-y^2\}$, defined by the series in the positive oval, which enters into the above equality. The function defined by the same series in the negative oval we shall denote by F_2.

If we put

$$2\alpha+2\beta=n+1-m,\ 2\alpha+2\beta'=-m-n,\ 1-\alpha'+\alpha=1-m,$$

so that $2\alpha+2\alpha'+2\beta+2\beta'=1$, the formula to be proved may be written

$$F_1\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}$$
$$=F\{n+1-m,\ -m-n;\ 1-m;\ \tfrac{1}{2}(1-y)\}.$$

By the formula already quoted in § 6 we have

$$\frac{\Gamma\{\frac{1}{2}(n+1-m)\}\,\Gamma\{\frac{1}{2}(-m-n)\}\,\Gamma\{\frac{1}{2}(1-m-n)\}\,\Gamma\{1-\frac{1}{2}(m-n)\}}{\Gamma(1-m)}$$
$$\times F_1\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}$$
$$=\frac{-1}{2\pi\iota}\int_L\Gamma(-s)\,\Gamma(\tfrac{1}{2}-s)\,\Gamma\{\tfrac{1}{2}(n+1-m)+s\}\,\Gamma\{\tfrac{1}{2}(-m-n)+s\}\,y^{2s}\,ds.$$

The integral is a valid expression of the series F_1 provided $|\arg y|<\pi$. It indicates that the function F_1 has a cross cut along some part of, or the whole, of the negative half of the real axis. (The cross cut is actually from $-\infty$ to -1, and so enters within the negative loop of the lemniscate). The contour L is drawn parallel to the imaginary axis, with, of course, suitable loops, if necessary, to ensure that positive

sequences of poles of the subject of integration be to the right of the contour and negative sequences to the left.

Now by the multiplication formula for the gamma function we have

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}} \Gamma(s)\,\Gamma(s+\tfrac{1}{2}).$$

Therefore

$$\frac{\Gamma(n+1-m)\,\Gamma(-m-n)}{\Gamma(1-m)}\, 2^{2m+1}\sqrt{\pi}$$

$$\times F_1\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}$$

$$= \frac{-1}{2\pi\iota}\int_L \Gamma(-s)\,2^s\Gamma\{\tfrac{1}{2}(n+1-m+s)\}\Gamma\{\tfrac{1}{2}(-m-n+s)\}\{1-(1-y)\}^s\,ds.$$

Now if the contour M of the integral be parallel to the imaginary axis with loops to ensure that positive sequences of poles of the subject of integration are to the right and negative sequences to the left of the contour

$$\Gamma(-s)\,(1-x)^s = -\frac{1}{2\pi\iota}\int_M \Gamma(\phi-s)\,\Gamma(-\phi)\,(-x)^\phi\,d\phi$$

provided $|\arg(-x)| < \pi$, and $|\arg(1-x| < \pi$.*

Therefore the previous expression, provided $|\arg(y-1)| < \pi$,

$$= \frac{1}{2\pi\iota}\int_L 2^s\,\Gamma\{\tfrac{1}{2}(n+1-m+s)\}\,\Gamma\{\tfrac{1}{2}(-m-n+s)\}\,ds$$

$$\times \frac{1}{2\pi\iota}\int_M \Gamma(\phi-s)\,\Gamma(-\phi)\,(y-1)^\phi\,d\phi.$$

Now if we take L and M to be the parallel contours of the figure (fig. 7), we may invert the order of integration, for the integrals concerned may, by a tedious though not difficult investigation, be proved to satisfy the necessary conditions.

We thus obtain

$$\frac{1}{2\pi\iota}\int_M d\phi\,\Gamma(-\phi)\,(y-1)^\phi \frac{1}{2\pi\iota}$$

$$\int_L \Gamma\{\tfrac{1}{2}(n+1-m+s)\}\Gamma\{\tfrac{1}{2}(-m-n+s)\}\Gamma\{\tfrac{1}{2}(\phi-s)\}\Gamma\{\tfrac{1}{2}(\phi-s+1)\}\frac{2^{\phi-1}}{\sqrt{\pi}}\,ds.$$

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* See the author's paper, *Quarterly Journal of Mathematics*, Vol. XXXVIII., p. 110

But I have previously (H. F. § 15) shown that

$$-\frac{1}{2\pi\iota}\int_L \Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,\Gamma(\beta_1-s)\,\Gamma(\beta_2-s)\,ds$$

$$=\frac{\Gamma(\alpha_1+\beta_1)\,\Gamma(\alpha_1+\beta_2)\,\Gamma(\alpha_2+\beta_1)\,\Gamma(\alpha_2+\beta_2)}{\Gamma(\alpha_1+\alpha_2+\beta_1+\beta_2)}.$$

The previous expression is therefore equal to

$$-\frac{1}{2\pi\iota}\,\frac{1}{\sqrt{\pi}}\int\Gamma(-\phi)\,2^{\phi}(y-1)^{\phi}\,d\phi$$

$$\times\frac{\Gamma\{\frac{1}{2}(\phi+n+1-m)\}\Gamma\{\frac{1}{2}(\phi+n-m)+1\}\Gamma\{\frac{1}{2}(\phi-m-n)\}\Gamma\{\frac{1}{2}(\phi-m-n)+\frac{1}{2}\}}{\Gamma(1-m+\phi)}$$

$$=-\frac{1}{2\pi\iota}\sqrt{(\pi)}\,2^{2m+1}\int\Gamma(-\phi)\frac{\Gamma(\phi+n-m+1)\,\Gamma(\phi-m-n)}{\Gamma(1-m+\phi)}\{\tfrac{1}{2}(y-1)\}^{\phi}d\phi$$

$$=\frac{\Gamma(n+1-m)\,\Gamma(-m-n)}{\Gamma(1-m)}\,2^{2m+1}\sqrt{\pi}$$

$$\times F\{n+1-m,\,-m-n;\;1-m;\;\tfrac{1}{2}(1-y)\}.$$

We thus have, provided $|\arg(y-1)|<\pi$,

$$F_1\{\tfrac{1}{2}(n+1-m),\,\tfrac{1}{2}(-m-n);\;1-m;\;1-y^2\}$$

$$=F\{n+1-m,\,-m-n;\;1-m;\;\tfrac{1}{2}(1-y)\},$$

which is the required result. The condition $|\arg(y-1)|<\pi$ can obviously be replaced by the narrower condition $|\arg y|<\pi$, since each series is one-valued when $0\leqq y\leqq 1$. Since the series on the right-hand side is convergent within a circle, radius 2 and centre $y=1$, we see that F_1 is uniform provided $|\arg(y+1)|<\pi$: F_1 has in fact a singularity when $y=-1$.

The formula may be written [H. F. § 11 (A)]

$$F_1\{\tfrac{1}{2}(n+1-m),\,\tfrac{1}{2}(-m-n);\;1-m;\;1-y^2\}$$

$$=\{\tfrac{1}{2}(1+y)\}^m F\{-n,\,n+1;\;1-m;\;\tfrac{1}{2}(1-y)\}.$$

The formula may be exhibited as the equality of two contour integrals. For by H.F. § 16 we have

$$-\frac{1}{2\pi\iota}\int\Gamma(-s)\,\Gamma(\rho-\alpha_1-\alpha_2-s)\,\Gamma(\alpha_1+s)\,\Gamma(\alpha_2+s)\,(1-x)^s\,ds$$

$$=\frac{\Gamma(\rho-\alpha_1)\,\Gamma(\rho-\alpha_2)\,\Gamma(\alpha_1)\,\Gamma(\alpha_2)}{\Gamma(\rho)}\,F\{\alpha_1,\,\alpha_2;\;\rho;\;x\}$$

where $|\arg(1-x)|<\pi$ if the contour be parallel to the imaginary axis.

Hence we have, when $|\arg x^2|<\pi$, $|\arg\frac{1}{2}(x+1)|<\pi$,

$$\int\Gamma(-s)\,\Gamma(\tfrac{1}{2}-s)\,\Gamma(\alpha+s)\,\Gamma(\beta+s)\,x^{2s}\,ds$$

$$=\cos(\alpha-\beta)\,2^{2-2\alpha-2\beta}$$

$$\int\Gamma(-s)\,\Gamma(\tfrac{1}{2}-\alpha-\beta-s)\,\Gamma(2\alpha+s)\,\Gamma(2\beta+s)\,\{\tfrac{1}{2}(1+x)\}^s\,ds,$$

the contours being parallel to the imaginary axis.

We note that the formula may be equally written

$$F\{2\alpha,\,2\beta;\ \gamma;\ \tfrac{1}{2}(1-y)\}=F_1\{\alpha,\,\beta;\ \gamma;\ 1-y^2\}$$

where $\gamma=\alpha+\beta+\frac{1}{2}$.

Put now $1-y^2=z$ so that $y=+\sqrt{(1-z)}$, since $R(y)$ is positive in F_1. Then we obtain, provided $|\arg(1-z)|<\pi$,

$$F\{\alpha,\,\beta;\ \gamma;\ z\}=F[2\alpha,\,2\beta;\ \gamma;\ \tfrac{1}{2}\{1-\sqrt{(1-z)}\}].$$

Putting $1-y=2x$, we obtain, if $R(x)<\frac{1}{2}$,

$$F\{2\alpha,\,2\beta;\ \gamma;\ x\}=F\{\alpha,\,\beta;\ \gamma;\ 4x(1-x)\}.$$

§ 16. The function

$$F_2\{\tfrac{1}{2}(n+1-m),\,\tfrac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}$$

can be transformed by writing $-y$ for y in the previous investigation. We obtain

$$F_2=\{\tfrac{1}{2}(1-y)\}^m F\{-n,\,n+1;\ 1-m;\ \tfrac{1}{2}(1+y)\},$$

provided $|1+y|<2$ and $|\arg(1-y)|<\pi$.

We have the interesting phenomenon that the differential equation for $F\{\frac{1}{2}(n+1-m),\ \frac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}$, which may be derived from Legendre's equation by an obvious transformation, has *both* its fundamental solutions represented by this series, each oval of convergence giving a different solution.

§ 17. As an immediate corollary from the previous result we may show that

$$F\{-n,\,n+1;\ 1-m;\ \tfrac{1}{2}\}=\frac{\Gamma\{\frac{1}{2}(1-m)\}\,\Gamma(1-\frac{1}{2}m)}{\Gamma\{1+\frac{1}{2}(n-m)\}\,\Gamma\{\frac{1}{2}(1-n-m)\}}.$$

For putting $y=0$ in the result of § 15 we have

$$F\{-n,\,n+1;\ 1-m;\ \tfrac{1}{2}\}=2^m F\{\tfrac{1}{2}(n+1-m),\,\tfrac{1}{2}(-m-n);\ 1-m;\ 1\},$$

and therefore the given result follows at once by Gauss' theorem. The formula may equally be obtained from the

theory of difference equations, and by its means we can give an alternative proof of the fundamental theorem of § 15.

§ 18. *The expression of* $P_n^m(x)$ *as hypergeometric series of arguments* $1-x^2$ *and* $1-1/x^2$.

The result of § 15 coupled with the definition of § 5 at once gives us the series

$$P_n^m(x)=\frac{2^m(x^2-1)^{-\frac{1}{2}m}}{\Gamma(1-m)}F_1\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-x^2\}$$

which is valid where $|1-x^2|<1$, $R(x)>0$.

By Kummer's formula {H. F. § 11 (A)} we may equally write this in the form

$$P_n^m(x)=\frac{2^m(x^2-1)^{-\frac{1}{2}m}}{\Gamma(1-m)}x^{n+m}F\{\tfrac{1}{2}(1-m-n),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-1/x^2\}.$$

From the first of these series we see that we have the contour integral

$$P_n^m(x)=\frac{2^m}{\Gamma\{\frac{1}{2}(n+1-m)\}\,\Gamma\{\frac{1}{2}(-m-n)}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int\frac{\Gamma\{s+\frac{1}{2}(n+1)\}\,\Gamma(s-\frac{1}{2}n)\,\Gamma(-\frac{1}{2}m-s)}{\Gamma(1-\frac{1}{2}m+s)}(x^2-1)^s\,ds,$$

wherein $|\arg(x^2-1)|<\pi$, if the contour be parallel to the imaginary axis.

§ 19. *The expansion of* $P_n^m(x)$ *in terms of hypergeometric series of arguments* $1/(1-x^2)$ *and* $1/x^2$.

Reverse the previous integral in the usual manner, and we obtain, when $|1-x^2|>1$,

$$P_n^m(x)=\frac{2^m\Gamma(-n-\frac{1}{2})}{\Gamma\{\frac{1}{2}(-m-n)\}\,\Gamma\{\frac{1}{2}(1-n-m)\}}(x^2-1)^{-\frac{1}{2}(n+1)}$$

$$\times F\left\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(n+m+1);\ \tfrac{3}{2}+n;\ \frac{1}{1-x^2}\right\}$$

$$+\frac{2^m\Gamma(n+\frac{1}{2})}{\Gamma\{\frac{1}{2}(n+1-m)\}\,\Gamma\{1+\frac{1}{2}(n-m)\}}(x^2-1)^{\frac{1}{2}n}$$

$$\times F\left\{\tfrac{1}{2}(m-n),\ \tfrac{1}{2}(-m-n);\ \tfrac{1}{2}-n;\ \frac{1}{1-x^2}\right\},$$

which is the first of the required series.

Using Kummer's formula just quoted and we find, when $|x|>1$,

$$P_n^m(x)\frac{\sqrt{\pi}\,x^m}{(x^2-1)^{\frac{1}{2}m}}=\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1-m)}(2x)^n \times F\{\tfrac{1}{2}(m-n),\ \tfrac{1}{2}(m-n+1);\ \tfrac{1}{2}-n;\ 1/x^2\}$$

$$+\frac{\Gamma(-n-\frac{1}{2})}{\Gamma(-m-n)}(2x)^{-n-1}F\{\tfrac{1}{2}(m+n)+1,\ \tfrac{1}{2}(m+n+1);\ n+\tfrac{3}{2};\ 1/x^2\}$$

to which Hobson's formula (22, p. 463) can be reduced.

Each of these series has only been established under the limitation $|\arg(x^2-1)|<\pi$. But the first series are convergent for the area outside the lemniscate of § 15, and the latter outside the circle $|x|=1$. Moreover, $P_n^m(x)$ is, as has been seen in § 13, a uniform analytic function within these areas provided they be dissected by a cross cut from $-\infty$ to -1. Therefore the formula will be valid when $|\arg x|<\pi$, provided we take the analytic continuations of the functions $(x^2-1)^{-\frac{1}{2}(n+1)}$ and $(x^2-1)^{\frac{1}{2}n}$. Each of these functions must therefore be such that $(x^2-1)^\alpha=(x-1)^\alpha(x+1)^\alpha$ where $|\arg(x\pm1)|<\pi$.

§ 20. *The expansion of $P_n^m(x)$ when $|x|<1$.*

We have just seen that, when $R(x)>0$,

$$P_n^m(x)(x^2-1)^{\frac{1}{2}m}2^{-m}=\frac{1}{\Gamma(1-m)} \times F\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-x^2\}.$$

Hence (by H. F. § 16), when $|\arg(x^2-1)|<\pi$,

$$P_n^m(x)(x^2-1)^{\frac{1}{2}m}\pi\,\Gamma(-m-n)\,\Gamma(1+n-m)\,2^{m+1} = \frac{-1}{2\pi\iota}\int\Gamma(-s)\,\Gamma(\tfrac{1}{2}-s)\,\Gamma\{\tfrac{1}{2}(n+1-m)+s\}\,\Gamma\{\tfrac{1}{2}(-m-n)+s\}\,x^{2s}\,ds.$$

Hence, when $|x|<1$, we have

$$P_n^m(x)(x^2-1)^{\frac{1}{2}m}2^{-m}=\frac{\Gamma(\frac{1}{2})}{\Gamma\{\frac{1}{2}(n-m)+1\}\,\Gamma\{\frac{1}{2}(1-m-n)\}} \times F\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ \tfrac{1}{2};\ x^2\}$$

$$+\frac{\Gamma(-\frac{1}{2})}{\Gamma\{\frac{1}{2}(n+1-m)\}\,\Gamma\{\frac{1}{2}(-m-n)\}}\,x\,F\{\tfrac{1}{2}(n-m)+1,\ \tfrac{1}{2}(1-m-n);\ \tfrac{3}{2};\ x^2\},$$

wherein, of course, $\arg(x^2-1)$ is equal to $\arg(x-1)+\arg(x+1)$ and $|\arg(x\pm1)|<\pi$. Hobson's more complicated formula (27, p. 469) can be reduced to that just given.

Evidently any other formula expressing $P_n^m(x)$ in hypergeometric series of arguments derivable by homographic transformation from x^2 can now be written down at once.

§ 21. *A series for $Q_n^m(x)$ when $|x|>1$.*

We can now show that, when $|x|>1$,

$$Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}=\frac{\Gamma(n+m+1)\,\Gamma(\frac12)}{2^{n+1}\Gamma(n+\frac32)}\frac{(x^2-1)^{\frac12 m}}{x^{n+m+1}}$$
$$\times F\{\tfrac12(n+m)+1\,;\ \tfrac12(n+m)+\tfrac12\,;\ n+\tfrac32\,;\ 1/x^2\},$$

where $|\arg x|<\pi$, and $(x^2-1)^{\frac12 m}=(x-1)^{\frac12 m}(x+1)^{\frac12 m}$, where $|\arg(x\pm1)|<\pi$ [H. 10, p. 456].

We have (§ 9)

$$2Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}=\Gamma(m)\left(\frac{x+1}{x-1}\right)^{\frac12 m}F\{-n,n+1;\ 1-m;\ \tfrac12(1-x)\}$$
$$+\frac{\Gamma(-m)\Gamma(1+m+n)}{\Gamma(1+n-m)}\left(\frac{x-1}{x+1}\right)^{\frac12 m}F\{-n,\ n+1;\ 1+m;\ \tfrac12(1-x)\},$$

And by § 15 this expression is equal to

$$\Gamma(m)\frac{2^m}{(x^2-1)^{\frac12 m}}F_1\{\tfrac12(n+1-m),\ \tfrac12(-m-n)\,;\ 1-m\,;\ 1-x^2\}$$
$$+\frac{\Gamma(-m)\,\Gamma(1+n+m)}{\Gamma(1+n-m)}2^{-m}(x^2-1)^{\frac12 m}$$
$$\times F_1\{\tfrac12(n+1+m),\ \tfrac12(m-n)\,;\ 1+m\,;\ 1-x^2\}.$$

Putting $1-x^2=y$ we therefore have, if $|\arg(-y)|<\pi$,

$$Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}\frac{2^{1-m}\Gamma\{\frac12(n-m+1)\}}{\Gamma\{\frac12(n+m)+1\}}(-y)^{\frac12(n+1)}$$
$$=-\left(\frac{1}{2\pi i}\right)\int\frac{\Gamma\{\frac12(n+m+1)+s\}\Gamma\{\frac12(n-m+1)+s\}\Gamma(-s)}{\Gamma(n+\frac32+s)}(-y)^s\,ds,$$

for the last integral, when $|y|<1$, can be taken to embrace the negative sequences of zeroes, and gives rise to the two series just written down.

Hence

$$Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}\frac{2^{1-m}}{\Gamma\{\frac{1}{2}(n+m)+1\}}$$

$$=(-y)^{-\frac{1}{2}(n+1)}\frac{\Gamma\{\frac{1}{2}(n+m+1)\}}{\Gamma(n+\frac{3}{2})}$$

$$\times F\{\tfrac{1}{2}(n+m+1),\tfrac{1}{2}(n-m+1);\ n+\tfrac{3}{2};\ 1/y\},$$

or

$$Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}=\frac{\Gamma(n+m+1)\,\Gamma(\frac{1}{2})}{2^{n+1}\,\Gamma(n+\frac{3}{2})}\frac{(x^2-1)^{\frac{1}{2}m}}{x^{n+m+1}}$$

$$\times\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}(n+m+1)}F\left\{\tfrac{1}{2}(n+m+1),\tfrac{1}{2}(n-m+1);\ n+\tfrac{3}{2};\ \frac{1}{1-x^2}\right\},$$

and this is equal to the given result [H. F. § 11(*A*)].

The final series

$$F\{\tfrac{1}{2}(n+m)+1,\tfrac{1}{2}(n+m)+\tfrac{1}{2};\ n+\tfrac{3}{2};\ 1/x^2\}$$

is convergent when $|x|>1$. We are therefore no longer restricted by the condition $|\arg(x^2-1)|<\pi$, but in the definition of $(x^2-1)^{\frac{1}{2}m}$ we must make it equal to

$$(x-1)^{\frac{1}{2}m}(x+1)^{\frac{1}{2}m}$$

where $|\arg(x\pm1)|<\pi$.

We note that incidentally we have established the transformation

$$F\{\tfrac{1}{2}(n+m)+1,\tfrac{1}{2}(n+m)+\tfrac{1}{2};\ n+\tfrac{3}{2};\ 1/x^2\}$$

$$=\left(\frac{x}{x-1}\right)^{m+n+1}F\left\{m+n+1,n+1;\ 2n+2;\ \frac{2}{1-x}\right\}.$$

This may of course be established directly.

The formula of the present paragraph gives us, when $|\arg(-x^2)|<\pi$, the contour integral

$$Q_n^m(x)=\frac{\sin(n+m)\pi}{\sin n\pi}\frac{(x^2-1)^{\frac{1}{2}m}\sqrt{\pi}}{2^{n+1}x^{n+m+1}}$$

$$\times\left(-\frac{1}{2\pi i}\right)\int\frac{\Gamma(2s+n+m+1)\,\Gamma(-s)}{\Gamma(n+\frac{3}{2}+s)}\left(-\frac{1}{4x^2}\right)^s ds.$$

§ 22. *The expansion of* $Q_n^m(x)$, *when* $|x|<1$.

Take the previous contour integral and bend it round so as to include the negative sequences of poles. Then, since

$(-1/x^2)^{-\alpha} = x^{2\alpha} e^{\mp \alpha\pi\iota}$, the upper or lower sign being taken as $I(x)$ is positive or negative, we have

$$Q_n^m(x) \frac{\sin n\pi}{\sin(n+m)\pi} \frac{(x^2-1)^{-\frac{1}{2}m}}{2^m\sqrt{\pi}}$$

$$= \frac{\Gamma\{\frac{1}{2}(n+m)+1\}}{\Gamma\{\frac{1}{2}(n-m+1)\}} e^{\mp(n+m)\frac{1}{2}\pi\iota} xF\{\tfrac{1}{2}(n+m)+1,\ \tfrac{1}{2}(1-n+m);\ \tfrac{3}{2};\ x^2\}$$

$$+ \frac{\Gamma\{\frac{1}{2}(n+m+1)\}}{2\,\Gamma\{\frac{1}{2}(n-m)+1\}} e^{\mp(n+m+1)\frac{1}{2}\pi\iota} F\{\tfrac{1}{2}(n+m+1),\ \tfrac{1}{2}(m-n);\ \tfrac{1}{2};\ x^2\}.$$

This formula embraces the two formulæ obtained by Hobson {(25), p. 467, and (26), p. 468}. In it

$$\arg(x^2-1) = \arg(x-1) + \arg(x+1)$$

where $|\arg(x\pm 1)| < \pi$. We have only proved this formula under the limitation $|\arg(-x^2)| < \pi$. That this restriction may be removed is evident from the considerations introduced at the end of § 19, or may be at once established analytically by means of the formula (§ 10)

$$Q_n^m(-x) = -e^{\pm n\pi\iota} Q_n(x).$$

Evidently any other formula expressing $Q_n^m(x)$ in hypergeometric series of arguments derivable by homographic transformation from x^2 can now be at once obtained.

Expansions of $P_n^m(x)$ and $Q_n^m(x)$ in series of arguments derived by homographic transformation from ξ^2, where $\xi = x - \sqrt{(x^2-1)}$ and $|\xi| < 1$.

§ 23. We now proceed to connect $P_n^m(x)$ and $Q_n^m(x)$ with the 24 series derivable by homographic transformation from $\{x-\sqrt{(x^2-1)}\}^2$. We thus complete the Riemann theory mentioned in § 3.

For adequate definition of our procedure we must briefly discuss the transformation

$$\xi = x - \sqrt{(x^2-1)}.$$

In this equality we assume that such a value of $\sqrt{(x^2-1)}$ is taken that $|\xi| < 1$.

We have $1/\xi = x + \sqrt{(x^2-1)}$, so that

$$2x = \xi + 1/\xi, \quad 2\sqrt{(x^2-1)} = 1/\xi - \xi.$$

If we put $\xi = re^{\iota\theta}$ and $x = x' + \iota y'$, we have

$$2x' = (r + 1/r)\cos\theta,$$
$$2y' = (r - 1/r)\sin\theta.$$

Hence the interior of the circle $|\xi| = 1$ maps on to the whole of the x-plane except for the cross cut from -1 to 1. This cross cut corresponds to the circle $|\xi| = 1$. A circle in the ξ-plane corresponds to an ellipse, foci ± 1 in the x-plane.

The positive half of the x-plane corresponds to the positive half of the ξ-plane. The two points ξ and $1/\xi$ correspond to the same point in the x-plane.

Again $$\arg x = -\tan^{-1}\left\{\frac{1/r - r}{1/r + r}\tan\theta\right\},$$

and therefore $|\arg(x\xi)| < \frac{1}{2}\pi$ when $|\xi| < 1$.

Hence $x^\alpha \xi^\alpha = (x\xi)^\alpha$, where α is any complex quantity and the principal values of the two expressions, *i.e.* values such that $|\arg x| < \pi$, $|\arg\xi| < \pi$, $|\arg(x\xi)| < \pi$, are taken.

Moreover, *that value of* $\sqrt{(x^2-1)}$ *which intervenes in the formula*

$$\xi = x - \sqrt{(x^2 - 1)}$$

is, when $|\xi| < 1$, *equal to* $\sqrt{(x-1)}\sqrt{(x+1)}$, *where*

$$|\arg(x+1)| < \pi, \quad |\arg(x-1)| < \pi.$$

For $$2\sqrt{(x^2-1)} = 2(x - \xi)$$
$$= (1/r - r)\cos\theta - \iota(r + 1/r)\sin\theta$$

Therefore, when $|\xi| = r < 1$, we have

$$|\arg\sqrt{(x^2-1)}| < \tfrac{1}{2}\pi \text{ when } R(x) > 0,$$

$\arg\sqrt{(x^2-1)}$ lying between $\frac{1}{2}\pi$ and π when x is in the second quadrant and $\arg\sqrt{(x^2-1)}$ lying between $-\frac{1}{2}\pi$ and $-\pi$ when x is in the third quadrant.

In all cases therefore $\sqrt{(x^2-1)} = \sqrt{(x-1)}\sqrt{(x+1)}$ with the given specification.

§ 24. We can now show that, if $|x| > 1$, $|\xi| < 1$,

$$F\{\tfrac{1}{2}(n+m)+1,\ \tfrac{1}{2}(n+m)+\tfrac{1}{2};\ n+\tfrac{3}{2};\ 1/x^2\}$$
$$= (2x\xi)^{n+m+1} F\{n+m+1,\ m+\tfrac{1}{2};\ n+\tfrac{3}{2};\ \xi^2\},$$

where $|\arg x\xi| < \pi$.

This theorem is fundamental in transformations to series of arguments derivable by homographic substitution from ξ^2

It may be proved by exactly the same process as the corresponding theorem established in § 15. We proceed to show that it reduces to that theorem by a simple change of the variable. The more general transformation of Riemann's P-function which includes both the present theorem and that of § 15 is proved in § 36 *infra*.

In § 15 we have see that

$$F\{1-m+n, -m-n;\ 1-m;\ \tfrac{1}{2}(1-y)\}$$
$$= F_1\{\tfrac{1}{2}(n+1-m), \tfrac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}.$$

Hence by an obvious substitution for m and n

$$F\{n+m+1, n-m+1;\ n+\tfrac{3}{2};\ \tfrac{1}{2}(1-y)\}$$
$$= F_1\{\tfrac{1}{2}(n+1+m), \tfrac{1}{2}(n-m+1);\ n+\tfrac{3}{2};\ 1-y^2\}.$$

Now $F\{\alpha_1, \alpha_2;\ \rho;\ x\} = (1-x)^{-\alpha_1} F\left\{\alpha_1, \rho-\alpha_2;\ \rho;\ \frac{x}{x-1}\right\}$

where $|\arg(1-x)| < \pi$.

We therefore have, if $|\arg(1+y)|$ and $|\arg y^2|$ are both less than π,

$$F\left\{n+m+1, \tfrac{1}{2}+m;\ n+\tfrac{3}{2};\ \frac{y-1}{y+1}\right\}$$
$$= \{\tfrac{1}{2}(1+y)\}^{n+m+1} (y^2)^{-\frac{1}{2}(n+m+1)}$$
$$\times F_1\{\tfrac{1}{2}(n+m+1), \tfrac{1}{2}(n+m)+1;\ n+\tfrac{3}{2};\ 1-1/y^2\}.$$

Thus, if $R(y) > 0$,

$$F\left\{n+m+1, \tfrac{1}{2}+m;\ n+\tfrac{3}{2};\ \frac{y-1}{y+1}\right\}$$
$$= \left(\frac{1+y}{2y}\right)^{n+m+1} F\{\tfrac{1}{2}(n+m+1), \tfrac{1}{2}(n+m)+1;\ n+\tfrac{3}{2};\ 1-1/y^2\},$$

where $\left|\arg\frac{1+y}{y}\right| < \frac{1}{2}\pi$.

Put now $$y = \frac{x}{\sqrt{(x^2-1)}},$$

where $|x| > 1$, and the square root is so chosen that $R(y) > 0$. We then have

$$\frac{1}{x^2} = 1 - \frac{1}{y^2}, \quad \frac{y-1}{y+1} = \frac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}} = \xi^2, \text{ where } |\xi| < 1,$$

and we also have $2y/(1+y) = 2x\xi$.

The last result may therefore be written, when $|x|>1$ and $|\xi|<1$,

$$F\{\tfrac{1}{2}(n+m+1),\ \tfrac{1}{2}(n+m)+1;\ n+\tfrac{3}{2};\ 1/x^2\}$$
$$=(2x\xi)^{n+m+1}F\{n+m+1,\ m+\tfrac{1}{2};\ n+\tfrac{3}{2};\ \xi^2\},$$

wherein $|\arg x\xi|<\tfrac{1}{2}\pi$.

The last condition $|\arg(x\xi)|<\tfrac{1}{2}\pi$ does not really impose limitations which at first sight might be suspected. For by § 23

$$2x\xi=1+r^2\cos 2\theta+ir^2\sin 2\theta,$$

and therefore, if $|\xi|<1$, $|\arg x\xi|$ is always less than $\tfrac{1}{2}\pi$. We have then in $(2x\xi)^{n+m+1}$ merely to take the principal value of the expression, and the final theorem can be stated in the form given in the enunciation.

Had we taken the equality (§ 16)

$$F_1\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ 1-m;\ 1-y^2\}$$
$$=\{\tfrac{1}{2}(1-y)\}^m F\{-n,\ n+1;\ 1-m;\ \tfrac{1}{2}(1+y)\}$$

valid when $R(y)<0$, we should have worked with the transformation

$$y=-\frac{x}{\sqrt{(x^2-1)}}=-\frac{x}{x-\xi}$$

where $|\xi|<1$, and we should have obtained the same result.

§ 25. *The expansion of $Q_n^m(x)$ in powers of ξ^2, when $|\xi|<1$.*

We will now show that when $|\xi|<1$, $|\arg\xi|<\pi$, $|\arg(1-\xi^2)|<\pi$ [H. (35), p. 475],

$$Q_n^m(x)=\frac{\Gamma(\tfrac{1}{2})\,\Gamma(-\tfrac{1}{2}-n)}{\Gamma(-n-m)}\cot n\pi\,(1-\xi^2)^m\,\xi^{n+1}$$
$$\times F\{m+\tfrac{1}{2},\ n+m+1;\ n+\tfrac{3}{2};\ \xi^2\}.$$

We have seen that, when $|x|>1$ (§ 21),

$$Q_n^m(x)=\frac{\sin(m+n)\pi}{\sin n\pi}\,\frac{\Gamma(n+m+1)\,\Gamma(\tfrac{1}{2})}{2^{n+1}\Gamma(n+\tfrac{3}{2})}\,\frac{(x^2-1)^{\frac{1}{2}m}}{x^{n+m+1}}$$
$$\times F\{\tfrac{1}{2}(n+m)+1,\ \tfrac{1}{2}(n+m)+\tfrac{1}{2};\ n+\tfrac{3}{2};\ 1/x^2\},$$

wherein $|\arg x|<\pi$ and $(x^2-1)^{\frac{1}{2}m}=(x-1)^{\frac{1}{2}m}(x+1)^{\frac{1}{2}m}$, where $\arg(x\pm1)|<\pi$.

Now by § 23, when $|\xi|<1$,

$$(x\xi)^{n+m+1}=x^{n+m+1}\,\xi^{n+m+1},$$

where $|\arg x\xi|<\frac{1}{2}\pi$, $|\arg x|<\pi$, $|\arg\xi|<\pi$.

If then we combine the previous result with that of § 24, we have

$$Q_n^{\,m}(x)=\cot n\pi\,\frac{\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}-n)}{2^{n+1}\,\Gamma(-n-m)}\,(x^2-1)^{\frac{1}{2}m}(2\xi)^{n+m+1}$$
$$\times F\{n+m+1,\ m+\tfrac{1}{2};\ n+\tfrac{3}{2};\ \xi^2\}.$$

Now, by § 23,

$$\frac{1-\xi^2}{2\xi}=\sqrt{(x-1)}\,\sqrt{(x+1)},$$

wherein $|\arg(x\pm1)|<\pi$.

Therefore $(1-\xi^2)^m=(2\xi)^m(x^2-1)^{\frac{1}{2}m}$ where $|\arg(1-\xi^2)|<\pi$.

The last result may therefore be written in the form given.

From this result we see that we have the contour integral expression

$$Q_n^{\,m}(x)\,\frac{\sin n\pi}{\sin(n+m)\,\pi}\,\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$
$$=-\frac{1}{2\pi\iota}\,(1-\xi^2)^m\,\xi^{n+1}\int\frac{\Gamma(n+m+1+s)\,\Gamma(m+\frac{1}{2}+s)\,\Gamma(-s)}{\Gamma(n+\frac{3}{2}+s)}(-\xi^2)^s\,ds.$$

The contour may be taken parallel to the imaginary axis (with of course suitable loops if necessary) provided $|\arg\xi|<\pi$ and $|\arg(-\xi^2)|<\pi$.

The integral will represent the analytic continuation of $Q_n^{\,m}(x)$ when $|\xi|>1$, provided $(1-\xi^2)^m=(1-\xi)^m(1+\xi)^m$, wherein $|\arg(1\pm\xi)|<\pi$.

§ 26. *Expression of $Q_n^{\,m}(x)$ as a hypergeometric series of argument $\xi^2/(\xi^2-1)$.*

By Kummer's formula [H. F. § 11 (*A*)] we have at once, if $|\arg(1-\xi^2)|<\pi$,

$$Q_n^{\,m}(x)=\cot n\pi\,\frac{\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}-n)}{\Gamma(-n-m)}\,\xi^{n+1}(1-\xi^2)^{-\frac{1}{2}}$$
$$\times F\{m+\tfrac{1}{2},\ \tfrac{1}{2}-m;\ n+\tfrac{3}{2};\ \xi^2/(\xi^2-1)\}.$$

The series is convergent within that area in the ξ-plane which includes the origin and lies between the branches o

a rectangular hyperbola of major axis $\sqrt{2}$. In the ξ-plane the analytic continuation of $Q_n^m(x)$ outside the circle $|\xi|=1$ will be given by the same series for all points ξ within the area of convergence, provided that $(1-\xi^2)^{-\frac{1}{2}}=(1-\xi)^{-\frac{1}{2}}(1+\xi)^{-\frac{1}{2}}$, where $|\arg(1\pm\xi)|<\pi$. For, with this restriction, the expression on the right-hand side of the formula is an analytic function within the area of convergence. If, then, we suppose that $Q_n^m(x)$ is continued across $|\xi|=1$ in the ξ-plane, so as to take values which correspond to its continuation across the cut $(-1, 1)$ in the x-plane, the equality will continue to hold. But it must be distinctly understood that the x-plane is a two-sheeted Riemann surface, and that the value of $Q_n^m(x)$ in the ξ-plane at a point outside $|\xi|=1$ is not its value at the point x in the original sheet of the x-plane which corresponds to this point ξ.

Hobson's result [(57), p. 485] is equivalent to that just given though its form is more complicated.

§ 27. *Expansion of $Q_n^m(x)$ in powers of $(1-\xi^2)$.*

From the result of § 25 coupled with the formula already quoted at the end of § 15 we have, if $|\arg(1-\xi^2)|<\pi$,

$$Q_n^m(x)\frac{\sqrt{(\pi)}\sin n\pi}{\cos m\pi\sin(n+m)\pi}\Gamma(1+n-m)$$

$$=-\frac{1}{2\pi\iota}\xi^{n+1}\int\Gamma(m-s)\Gamma(-m-s)\Gamma(\tfrac{1}{2}+s)\Gamma(n+1+s)(1-\xi^2)^s\,ds.$$

Hence, if $|1-\xi^2|<1$, we have by Cauchy's theorem

$$-Q_n^m(x)\frac{\sin n\pi}{\pi}\Gamma(-n-m)=2^{-2m-1}\frac{\Gamma(-m)}{\Gamma(1+n-m)}(1-\xi^2)^m\xi^{n+1}$$
$$\times F\{m+\tfrac{1}{2},\, n+m+1;\ 1+2m;\ 1-\xi^2\}$$
$$+2^{2m-1}\frac{\Gamma(m)}{\Gamma(1+n+m)}(1-\xi^2)^{-m}\xi^{n+1}F\{-m+\tfrac{1}{2},\, n-m+1;\ 1-2m;\ 1-\xi^2\};$$

and by reversing the contour of the integral we obtain an analogous expression in powers of $1/(1-\xi^2)$. If we suppose that the function $Q_n^m(x)$ is continued when $|\xi|>1$ across the cut $(-1, 1)$ in the two-sheeted x-plane on to the second sheet, the expression just given holds for all values of ξ within the lemniscate area of convergence provided in ξ^{n+1} we have $|\arg\xi|<\pi$, and in $(1-\xi^2)^{\pm m}$ we have $|\arg(1-\xi^2)|<\pi$.

These conditions give cross cuts in the ξ-plane from $\xi = \pm 1$ in directions along the real axis away from the origin. As a verification we see that, if we replace $Q_n^m(x)$ by $G_n^m(\xi)$ in order to indicate the new variable,

$$G_n^m(-\xi) = e^{\mp(n+1)\pi i} G_n^m(\xi),$$

the upper or lower sign being taken as $I(\xi)$ is $\pm$, or $I(x) \mp$, which is evidently equivalent to the old relation of § 10.

By Kummer's relation [H. F. § 11 (A)] we have, if $R(\xi) > 0$,

$$F\{m + \tfrac{1}{2}, n + m + 1;\ 1 + 2m;\ 1 - \xi^2\}$$
$$= (\xi^2)^{-n-\frac{1}{2}} F\{\tfrac{1}{2} + m,\ m - n;\ 1 + 2m;\ 1 - \xi^2\}.$$

Hence, if $R(\xi) > 0$,

$$-2\, Q_n^m(x) \frac{\sin n\pi}{\pi} \Gamma(-n-m)$$
$$= 2^{-2m} \frac{\Gamma(-m)}{\Gamma(1+n-m)} (1-\xi^2)^m \xi^{-n} F\{\tfrac{1}{2}+m,\ m-n;\ 1+2m;\ 1-\xi^2\}$$
$$+ 2^{2m} \frac{\Gamma(m)}{\Gamma(1+n+m)} (1-\xi^2)^{-m} \xi^{-n} F\{\tfrac{1}{2}-m,\ -m-n;\ 1-2m;\ 1-\xi^2\}.$$

This formula is not valid when $R(\xi) < 0$. In this case we have

$$-2\, Q_n^m(x) \frac{\sin n\pi}{\pi} \Gamma(-n-m)\, e^{\mp(2n+1)\pi i}$$

equal to the sum of the two series just written down, the upper or lower sign being taken as $I(\xi)$ is positive or negative.

§ 28. *Expansion of $P_n^m(x)$ in powers of $1 - \xi^2$.*

We proceed to show that, if $R(\xi) > 0$, and $|1 - \xi^2| < 1$,

$$P_n^m(x) = \frac{2^{2m}}{\Gamma(1-m)} (1-\xi^2)^{-m} \xi^{n+1} F\{\tfrac{1}{2}-m,\ 1+n-m;\ 1-2m;\ 1-\xi^2\},$$

wherein in ξ^{n+1} we have $|\arg \xi| < \pi$ and in $(1-\xi^2)^{-m}$ we have $|\arg(1-\xi^2)| < \pi$. This expansion is not given by Hobson.

By § 12

$$P_n^m(x) \cos n\pi = \frac{\sin n\pi}{\pi} Q_n^m(x) - \frac{\sin n\pi}{\pi} Q^m{}_{-n-1}(x).$$

Hence, when $R(\xi)>0$, we have, by § 27,

$$P_n^m(x)\cos n\pi = -2^{2m-1}\frac{\Gamma(m)}{\Gamma(1+n+m)\,\Gamma(-n-m)}(1-\xi^2)^{-m}\xi^{n+1}$$
$$\times F\{-m+\tfrac{1}{2},\ n-m+1;\ 1-2m;\ 1-\xi^2\}$$
$$+2^{2m-1}\frac{\Gamma(m)}{\Gamma(1+n-m)\,\Gamma(-n+m)}(1-\xi^2)^{-m}\xi^{n+1}$$
$$\times F\{-m+\tfrac{1}{2},\ n-m+1;\ 1-2m;\ 1-\xi^2\},$$

for two series cancel.

Thus

$$P_n^m(x)\cos n\pi =$$
$$2^{2m-1}\frac{\Gamma(m)}{\Gamma(1+n-m)\,\Gamma(m-n)}\left\{1+\frac{\sin(m+n)\pi}{\sin(m-n)\pi}\right\}(1-\xi^2)^{-m}\xi^{n+1}$$
$$\times F\{-m+\tfrac{1}{2},\ n-m+1;\ 1-2m;\ 1-\xi^2\}$$
$$=\frac{2^{2m}\cos n\pi}{\Gamma(1-m)}(1-\xi^2)^{-m}\xi^{n+1}F\{\tfrac{1}{2}-m,\ 1+n-m;\ 1-2m;\ 1-\xi^2\}.$$

To obtain the expansion of $P_n^m(x)$ in powers of $(1-\xi^2)$, when $R(\xi)<0$, we may either use the same process and employ the final series of § 27, or we may use the relation (§ 10)

$$P_n^m(-x)=e^{\mp n\pi\iota}P_n^m(x)-\frac{2\sin n\pi}{\pi}Q_n^m(x).$$

We find, when $R(\xi)<0$,

$$P_n^m(x)=\frac{\sin n\pi}{\pi}\Gamma(m)\,2^{2m}e^{\mp(m+n)\pi\iota}(1-\xi^2)^{-m}\xi^{n+1}$$
$$\times F\{\tfrac{1}{2}-m,\ 1+n-m;\ 1-2m;\ 1-\xi^2\}$$
$$-\frac{\Gamma(-m)}{\Gamma(1+n-m)\,\Gamma(-m-n)}2^{-2m}e^{\mp n\pi\iota}(1-\xi^2)^{m}\xi^{n+1}$$
$$\times F\{\tfrac{1}{2}+m,\ 1+n+m;\ 1+2m;\ 1-\xi^2\},$$

the upper or lower sign being taken as $I(\xi)$ is positive or negative. In $(1-\xi^2)^{\pm m}$ we have $1-\xi^2=(1-\xi)(1+\xi)$, where $|\arg(1\pm\xi)|<\pi$, so that $|\arg(1-\xi^2)|<\pi$, and in ξ^{n+1} we have $|\arg\xi|<\pi$.

When $R(\xi)>0$, we have equally by Kummer's relation

$$P_n^m(x)=\frac{2^{n+m}}{\Gamma(1-m)}(1-\xi^2)^{-m}\xi^{-n-m}F\{\tfrac{1}{2}-m,\ -m-n;\ 1-2m;\ 1-\xi^2\}.$$

We see that, when $R(\xi)>0$, we have the contour integral

$$P_n^m(x)=\frac{1}{\Gamma(\frac{1}{2})\Gamma(1+n-m)}(1-\xi^2)^{-m}\xi^{n+m}$$

$$\times\left(\frac{-1}{2\pi\iota}\right)\int\frac{\Gamma(s+\tfrac{1}{2}-m)\,\Gamma(s+1+n-m)\,\Gamma(-s)}{\Gamma(1-2m+s)}(\xi^2-1)^s\,ds,$$

wherein $|\arg(\xi^2-1)|<\pi$ when the contour of the integral is parallel to the imaginary axis.

Of course we must always remember that, in any of these formulæ, when $|\xi|>1$, we pass to the continuation of $P_n^m(x)$ in the second sheet of the ξ-plane.

§ 29. *The expansion of $P_n^m(x)$ in powers of ξ^2.*

By the previous paragraph coupled with the result of H. F. § 16 we see that, provided $|\arg\xi^2|<\pi$.

$$P_n^m(x)\,\Gamma(\tfrac{1}{2})\,\Gamma(1+n-m)\,\Gamma(\tfrac{1}{2}-m)\,\Gamma(-m-n)$$

$$=-\frac{1}{2\pi\iota}(1-\xi^2)^{-m}\xi^{n+1}\int\Gamma(-s)\Gamma(-\tfrac{1}{2}-n-s)\Gamma(\tfrac{1}{2}-m+s)\Gamma(1+n-m+s)\xi^{2s}\,ds.$$

Hence, by Cauchy's theorem, when $R(\xi)>0$, $|\xi|<1$,

$$\sqrt{\pi}\,P_n^m(x)=\frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(-m-n)}(1-\xi^2)^{-m}\xi^{n+1}F\{\tfrac{1}{2}-m,\ 1+n-m;\ \tfrac{3}{2}+n;\ \xi^2\}$$

$$+\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1-m)}(1-\xi^2)^{-m}\xi^{-n}F\{\tfrac{1}{2}-m,\ -n-m;\ \tfrac{1}{2}-n;\ \xi^2\}.$$

By the argument already employed several times [cf. § 19] we see that this is valid when $|\xi|<1$ for all values of $R(\xi)$ provided we have the condition $|\arg\xi|<\pi$, and in $(1-\xi^2)^{-m}$ provided $|\arg(1-\xi^2)|<\pi$. When n is half an odd integer the expansion breaks down; the limiting form, which may be directly deduced from the contour integral, is somewhat complicated.

Since the two series change into one another when n is replaced by $-n-1$, we have the old relation

$$P^m(x)=P^m_{-n-1}(x).$$

Similarly by § 25 we see that we have again the relation

$$P_n^m(x) = Q_n^m(x)\frac{\tan n\pi}{\pi} - Q^m_{-n-1}(x)\frac{\tan n\pi}{\pi}.$$

The expansion of $P_n^m(x)$ in ascending powers of ξ may by Kummer's formula equally be written, when $|\xi|<1$,

$$\sqrt{\pi}\,P_n^m(x) = \frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(-m-n)}(1-\xi^2)^m\xi^{n+1}F\{\tfrac{1}{2}+m,\ 1+m+n;\ \tfrac{3}{2}+n;\ \xi^2\}$$

$$+\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1-m)}(1-\xi^2)^m\xi^{-n}F\{\tfrac{1}{2}+m,\ -n+m;\ \tfrac{1}{2}-n;\ \xi^2\},$$

wherein $|\arg\xi|<\pi$, $|\arg(1-\xi^2)|<\pi$.

Hobson's formula [(36), p. 475] may be reduced to this form.

§ 30. If we take the contour integral of § 29 the reverse way *we have the expansion of* $P_n^m(x)$ *in powers of* $1/\xi^2$.

We find, if $R(\xi)>0$ and $|\arg(1-\xi^2)|<\pi$, $|\xi|>1$,

$$P_n^m(x) = \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(1+n-m)}\xi^{2m+n}(1-\xi^2)^{-m}F\{\tfrac{1}{2}-m,\ -m-n;\ \tfrac{1}{2}-n;\ 1/\xi^2\}$$

$$+\frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(\frac{1}{2})\Gamma(-n-m)}\xi^{2m-n-1}(1-\xi^2)^{-m}F\{\tfrac{1}{2}-m,\ n+1-m;\ \tfrac{3}{2}+n;\ 1/\xi^2\}.$$

The condition $|\arg(1-\xi^2)|<\pi$ is obviously equivalent to making $(1-\xi^2)^m=(1-\xi)^m(1+\xi)^m$ where $|\arg(1\pm\xi)|<\pi$. The condition $R(\xi)>0$ can evidently be removed, and replaced by $|\arg\xi|<\pi$.

We proceed to investigate the meaning of this formula.

In the first place we see that the function $P_n^m(x)$, herein involved, is the continuation across the circle $|\xi|=1$ of the function previously defined when $|\xi|<1$. We have therefore passed over the cross cut $(-1, 1)$ in the x-plane, and thus we have passed to the second sheet of that plane, considered as a two-sheeted Riemann surface. We will denote the function at the point x in the original sheet by ${}_1P_n^m(x)$, and at the same point in the second sheet by ${}_2P_n^m(x)$.

Let us now put

$$F_n^m(\xi) = \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\Gamma(1+n-m)}\xi^{-n}(1-\xi^2)^{-m}F\{\tfrac{1}{2}-m,\ -n-m;\ \tfrac{1}{2}-n;\ \xi^2\}$$

$$+\frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(\frac{1}{2})\Gamma(-n-m)}\xi^{n+1}(1-\xi^2)^{-m}F\{\tfrac{1}{2}-m,\ 1+n-m;\ \tfrac{3}{2}+n;\ \xi^2\},$$

and postulate, as usual, that $(1-\xi^2)^m=(1-\xi)^m(1+\xi)^m$.

Then $F_n^m(\xi)$ is a function of ξ defined when $|\xi|<1$, and it may be analytically continued over the whole of the ξ-plane if the latter be dissected along the real axis except between $\xi=0$ and $\xi=1$.

We may write the relation just found in the form

$$_2P_n^m(x) = F_n^m(1/\xi)\, e^{\pm m\pi\iota}, \quad |\xi|>1,$$

the upper or lower sign being taken as $I(\xi)$ is positive or negative.

And the relation previously found may be written

$$_1P_n^m(x) = F_n^m(\xi), \quad |\xi|<1.$$

If now we pass from ξ, where $|\xi|<1$, across the circle $|\xi|=1$, $_1P_n^m(x)$ becomes $_2P_n^m(x)$; we therefore have, when $|\xi|>1$,

$$F_n^m(\xi) = F_n^m(1/\xi)\, e^{\pm m\pi\iota} \ldots\ldots\ldots\ldots\ldots\ldots (A),$$

the upper or lower sign being taken as $I(\xi)$ is positive or negative.

Changing ξ into $1/\xi$ we see that the same formula holds when $|\xi|<1$.

This relation may be verified by the following considerations:—

The function $F_n^m(\xi)$ is one-valued if the ξ-plane be dissected by a cross cut from $-\infty$ to 0 and 1 to ∞ along the real axis. In this dissected region we may go by a continuous path from ξ to $1/\xi$. This path corresponds to a complete circuit round the point $x=1$ in the x-plane. The figure (fig. 8) is drawn for the case when $I(x)$ is positive and $|\xi|<1$. In this case we see by § 6 that $P_n^m(x)$ is multiplied by $e^{m\pi\iota}$ in going round the circuit. Hence, as ξ and $1/\xi$ correspond to the same point in the x-plane, $F_n^m(\xi)$ is multiplied by $e^{m\pi\iota}$ when it becomes $F_n^m(1/\xi)$.

Thus, when $I(\xi)$ is negative and $|\xi|<1$,

$$F_n^m(1/\xi) = F_n^m(\xi)\, e^{m\pi\iota}.$$

The cases when $I(\xi)$ is positive or $|\xi|>1$ can be treated in the same way.

§ 31. *We may now obtain the expansion of* $Q_n^m(x)$ *in powers of* $1/\xi^2$, and show similarly that it corresponds to an obvious rotation-transformation in the x-plane.

Take the integral given at the end of § 25, reverse it and employ Cauchy's theorem, and we find if $|\xi|>1$, $|\arg\xi|<\pi$, $|\arg(-\xi^2)|<\pi$,

$$Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}$$

$$=(1-\xi^2)^m\xi^{n+1}(-\xi^2)^{-n-m-1}\frac{\Gamma(n+m+1)\,\Gamma(-n-\frac{1}{2})}{\Gamma(\frac{1}{2}-m)}$$

$$\times F\{n+m+1,\ \tfrac{1}{2}+m;\ \tfrac{3}{2}+n;\ 1/\xi^2\}$$

$$+(1-\xi^2)^m\xi^{n+1}(-\xi^2)^{-m-\frac{1}{2}}\frac{\Gamma(n+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(n-m+1)}$$

$$\times F\{m+\tfrac{1}{2},\ m-n;\ \tfrac{1}{2}-n;\ 1/\xi^2\}.$$

In this relation we must have $|\arg(1-\xi^2)|<\pi$ so that $(1-\xi^2)^m=(1-\xi)^m(1+\xi)^m$ where $|\arg(1\pm\xi)|<\pi$. The condition that $|\arg(-\xi^2)|<\pi$ may be interpreted to be $R(\xi)>0$, but ξ not real and positive. It may evidently be removed by the process of analytic continuation and replaced by the condition that the ξ-plane is dissected outside the circle $|\xi|=1$ by cross cuts along the two parts of the real axis. In thus continuing $Q_n^m(x)$ outside the circle $|\xi|=1$, x in its plane makes a circuit of the origin across the cross cut $(-1, 1)$ and we have to suppose that $Q_n^m(x)$ takes a value corresponding to this circuit. If the x-plane is a two-sheeted Riemann surface which corresponds to the whole ξ-plane and has winding points at $x=\pm1$, we shall have passed from a point x on one sheet to a point in the other sheet.

With these limitations the previous formula may be written

$$Q_n^m(x)\frac{\sin n\pi}{\sin(n+m)\pi}\frac{1}{\Gamma(\frac{1}{2})}(1-1/\xi^2)^m$$

$$=\Gamma(n+m+1)\,\Gamma(-n-\tfrac{1}{2})\frac{\cos m\pi}{\pi}e^{\pm(n+1)\pi\iota}\xi^{-n-1}$$

$$\times F\{\tfrac{1}{2}-m,\ 1+n-m;\ \tfrac{3}{2}+n;\ 1/\xi^2\}$$

$$\pm\iota\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+1)}\xi^nF\{-m+\tfrac{1}{2},\ -m-n;\ \tfrac{1}{2}-n;\ 1/\xi^2\},$$

wherein $|\xi|>1$, the upper or lower sign being taken as $I(\xi)$ is $\pm$.

§ 32. Put now

$$G_n^m(\xi) = \frac{\Gamma(\frac{1}{2})\,\Gamma(-\frac{1}{2}-n)}{\Gamma(-n-m)} \cot n\pi\,(1-\xi^2)^{-m}\,\xi^{n+1} \times F\{1+n-m,\ \tfrac{1}{2}-m;\ n+\tfrac{3}{2};\ \xi^2\},$$

when $|\xi|<1$, and suppose that $G_n^m(\xi)$, when $|\xi|>1$, is the analytic continuation of this series. The function then has a cross cut along the whole of the real axis except when ξ lies between 0 and 1.

Then, by § 25,

$$Q_n^m(x) = G_n^m(\xi) \text{ when } |\xi|<1.$$

The relation just found gives us, when $|\xi|>1$,

$$Q_n^m(x) = \frac{\cos m\pi}{\cos n\pi}\, e^{\pm n\pi\iota}\, G_n^m(1/\xi) \pm \iota \frac{\Gamma(\frac{1}{2})\sin(n+m)\pi}{\sin n\pi}\,(1-1/\xi^2)^{-m}\,\xi^n\,\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-m+1)} \times F\{-m+\tfrac{1}{2},\ -m-n;\ \tfrac{1}{2}-n;\ 1/\xi^2\}.$$

Now, if $F_n^m(\xi)$ be the function introduced in § 30, we have, when $|\xi|>1$,

$$F_n^m(1/\xi) = \frac{\tan n\pi}{\pi}\, G_n^m(1/\xi) + \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\,\Gamma(1+n-m)}\,\xi^n(1-1/\xi^2)^{-m} \times F\{-m+\tfrac{1}{2},\ -m-n;\ \tfrac{1}{2}-n;\ 1/\xi^2\}.$$

Therefore, when $|\xi|>1$,

$$Q_n^m(x) = e^{\mp m\pi\iota}\, G_n^m(1/\xi) \pm \iota\pi \frac{\sin(n+m)\pi}{\sin n\pi}\, F_n^m(1/\xi),$$

the upper or lower sign being taken as $I(\xi)$ is positive or negative.

Now, when ξ moves from ξ_0 ($|\xi_0|<1$) to ξ' ($|\xi'|>1$), without passing over the cross cuts in the ξ-plane along the real axis, $G_n^m(\xi_0)$ becomes $G_n^m(\xi')$.

Therefore

$$G_n^m(\xi') = e^{\mp m\pi\iota}\, G_n^m(1/\xi') \pm \iota\pi \frac{\sin(n+m)\pi}{\sin n\pi}\, F_n^m(1/\xi') \ldots (B),$$

the upper or lower sign being taken as $I(\xi')$ is $\pm$. This relation which has been established when $|\xi'|>1$ evidently holds also when $|\xi'|\leq 1$. It corresponds to the expansion

obtained at the end of the previous paragraph, and may be interpreted as follows:—

When we go from $1/\xi'$, $|\xi'|>1$, to ξ', the function $G_n^m(1/\xi') = Q_n^m(x)$ is changed owing to the description by x of a $\pm$ circuit round $x=1$ as $I(\xi')$ is $\mp$. Therefore corresponding to such a circuit $Q_n^m(x)$ becomes

$$e^{\pm m\pi\iota}\, Q_n^m(x) \mp \iota\pi \frac{\sin(n+m)\pi}{\sin n\pi} P_n^m(x).$$

This may be readily verified by our previous knowledge. For, by § 9, when x describes a $\pm$ circuit round $x=1$ (and not round $x=-1$), $Q_n^m(x)$ becomes

$$e^{\pm m\pi\iota}\, Q_n^m(x) + (e^{\mp m\pi\iota} - e^{\pm m\pi\iota}) \times \frac{\Gamma(m)}{\Gamma(1+n+m)} \frac{-\pi}{2\sin n\pi\,\Gamma(-m-n)} \Gamma(1-m) P_n^m(x)$$

$$= e^{\pm m\pi\iota}\, Q_n^m(x) \mp \iota\pi \frac{\sin(m+n)\pi}{\sin n\pi} P_n^m(x).$$

The verification is therefore complete.

§ 33. *When $-1<x<1$ it will evidently be useful to express the functions $P_n^m(x)$ and $Q_n^m(x)$, as we have defined them on the cross cut in §§ 13 and 14, in terms of the functions $F_n^m(\xi)$ and $G_n^m(\xi)$ which, as they have been defined in the previous paragraphs, have no cross cut when $|\xi|=1$.*

When $-1<x<1$ we put $x=\cos\theta$ where $0<\theta<\pi$.

Then $\xi = e^{\pm\iota\theta}$, and since by our definition the condition that $|\xi|$ is <1 necessitates that $I(x)$ and $I(\xi)$ always have opposite signs, we see that

if $x=\cos\theta+\varepsilon\iota$ where ε is very small and ultimately zero $\xi = e^{-\iota\theta}$,

if $x=\cos\theta-\epsilon\iota$ „ „ „ „ $\xi = e^{\iota\theta}$.

Now, by § 13,

$$P_n^m(\cos\theta) = e^{\frac{1}{2}m\pi\iota} P_n^m(\cos\theta+\epsilon\iota) = e^{-\frac{1}{2}m\pi\iota} P_n^m(\cos\theta-\varepsilon\iota).$$

Therefore

$$P_n^m(\cos\theta) = e^{\frac{1}{2}m\pi\iota} F_n^m(e^{-\iota\theta}) = e^{-\frac{1}{2}m\pi\iota} F_n^m(e^{\iota\theta}),$$

This agrees with the formula (A) of § 30, θ being positive, and $0<\theta<\pi$.

Again, by § 14, we shall have

$$Q_n^m(\cos\theta) = \tfrac{1}{2}\{Q_n^m(x+\epsilon\iota) + Q_n^m(x-\epsilon\iota)\}$$
$$= \tfrac{1}{2}\{G_n^m(e^{\iota\theta}) + G_n^m(e^{-\iota\theta})\};$$

and also

$$\frac{-\iota\pi\sin(m+n)\pi}{\sin n\pi} P_n^m(\cos\theta) = G_n^m(e^{-\iota\theta}) e^{-\frac{1}{2}m\pi\iota} - G_n^m(e^{\iota\theta}) e^{\frac{1}{2}m\pi\iota}.$$

This again agrees with the formula (B) of § 32.

§ 34. In the next place it will be convenient, when we come to consider the asymptotic expansions of $P_n^m(\cos\theta)$, $Q_n^m(\cos\theta)$, *to express* $P_n^m(-\cos\theta)$, $Q_n^m(-\cos\theta)$, *where* $0 < \theta \leq \frac{1}{2}\pi$, *in terms of our functions* F *and* G.

Since θ is positive, we have

$$Q_n^m(-\cos\theta + \epsilon\iota) = G_n^m(-e^{\iota\theta}) = G_n^m(e^{\iota\theta})\, e^{-(n+1)\pi\iota},$$

and
$$Q_n^m(-\cos\theta - \epsilon\iota) = G_n^m(-e^{-\iota\theta}) = G_n^m(e^{-\iota\theta})\, e^{(n+1)\pi\iota}.$$

We therefore have, when $0 < \theta < \frac{1}{2}\pi$,

$$Q_n^m(-\cos\theta) = \tfrac{1}{2}\{G_n^m(e^{\iota\theta})\, e^{-(n+1)\pi\iota} + G_n^m(e^{-\iota\theta})\, e^{(n+1)\pi\iota}\}.$$

And also, under the same condition,

$$\frac{-\iota\pi\sin(m+n)\pi}{\sin n\pi} P_n^m(-\cos\theta)$$
$$= G_n^m(e^{\iota\theta}) e^{-(n+1)\pi\iota - \frac{1}{2}m\pi\iota} - G_n^m(e^{-\iota\theta}) e^{(n+1)\pi\iota + \frac{1}{2}m\pi\iota}.$$

The general transformation of the second degree of Riemann's P-function.

§ 35. It has been stated in § 3 that the three sets of transformations of generalised Legendre functions are possible because they are substantially Riemann P-functions of a special type in which the difference of the two exponents associated with one of the singularities is equal to one-half. In this case a special transformation exists. Riemann obtained the form of this transformation for the case when the singularities of the P-function are $1, \infty, 0$ by general considerations. It is interesting to show that the analysis used in § 15 will give the complete form of the corresponding transformation for the P-function with general singularities a, b, c.

We will assume that, if a, b, c be represented by points A, B, C in the Argand diagram, these points are so placed that when we go round the circle through A, B, and C in

a positive counter-clockwise direction we pass from A to C to B. This, of course, is Riemann's fundamental assumption.

Further, as in my previous memior,* we will take one of the 24 P-functions, let us say that with suffix unity, so that we have

$$P_1\begin{Bmatrix} a, & b, & c, & \\ \alpha, & \beta, & \gamma, & x \\ \alpha', & \beta', & \gamma', & \end{Bmatrix}$$

$$= \frac{-1}{2\pi\iota}\left(\frac{x-c.b-a}{x-a.b-c}\right)^{\gamma}\int \frac{\Gamma(\alpha+\gamma-s)\,\Gamma(\beta+s)\,\Gamma(\beta'+s)}{\Gamma(1-\alpha'-\gamma+s)}\left(-\frac{x-a.c-b}{c-a.x-b}\right)^{s} ds$$

$$= \left(\frac{x-c.b-a}{x-b.c-a}\right)^{\gamma}\left(\frac{x-a.c-b}{c-a.x-b}\right)^{\alpha} e^{\mp\pi\iota\alpha}\,\frac{\Gamma(\alpha+\beta+\gamma)\,\Gamma(\alpha+\beta'+\gamma)}{\Gamma(1-\alpha'+\alpha)}$$

$$\times F\left\{\alpha+\beta+\gamma,\ \alpha+\beta'+\gamma;\ 1-\alpha'+\alpha;\ \frac{x-a.c-b}{c-a.x-b}\right\}.$$

We assume as before that $\dfrac{x-c.b-a}{b-c.x-a}$ has a cross cut along the arc AC, and that its argument ranges from π just within AC to zero on the arc ABC and to $-\pi$ just outside AC.

Similarly $\dfrac{x-a.c-b}{c-a.x-b}$ has a cross cut along BA, and $\dfrac{x-b.a-c}{a-b.x-c}$ along CB.

We also have the permanent relation

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1.$$

§ 36. *We can now show that, if* $\alpha+\alpha'+\beta+\beta'+2\gamma=\frac{1}{2}$,

$$P_1\begin{Bmatrix} a, & b, & c & , \\ \alpha, & \beta, & \gamma & , x \\ \alpha', & \beta', & \gamma+\frac{1}{2}, & \end{Bmatrix} = \frac{\pi 2^{2\alpha'+2\gamma+1}}{\Gamma(\alpha+\beta+\gamma+\frac{1}{2})\,\Gamma(\alpha+\beta'+\gamma+\frac{1}{2})}\left(\frac{x-c.b-a}{x-a.b-c}\right)^{\gamma}$$

$$\times P_1\left\{\begin{matrix} 1, & \infty, & -1 \\ \alpha+\gamma, & 2\beta, & \alpha+\gamma \\ \alpha'+\gamma, & 2\beta', & \alpha'+\gamma \end{matrix}\ \sqrt{\left(\frac{a-b.x-c}{a-c.x-b}\right)}\right\},$$

where $\sqrt{\left(\dfrac{a-b.x-c}{a-c.x-b}\right)}$ *is such that its argument is* $-\frac{1}{2}\pi$ *just inside the cross cut along the arc* CB, *is zero on the arc* BAC, *and is* $\frac{1}{2}\pi$ *just outside the cross cut* CB.

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* H. F. §§ 20–29.

Putting for brevity

$$u=\frac{x-b.a-c}{a-b.x-c},\quad v=\frac{x-c.b-a}{b-c.x-a},\quad w=\frac{x-a.c-b}{c-a.x-b},$$

we have [H. F. § 26], in the special case when $\gamma'-\gamma=\frac{1}{2}$,

$$w^{-\alpha}u^{\gamma}e^{\pm\pi i\alpha}\,\Gamma(\alpha+\beta'+\gamma')\,\Gamma(\alpha+\beta+\gamma')\,P_1\begin{Bmatrix}a, & b, & c, & \\ \alpha, & \beta, & \gamma, & x\\ \alpha', & \beta', & \gamma+\frac{1}{2}, & \end{Bmatrix}$$

$$=\frac{-1}{2\pi i}\int\Gamma(-s)\,\Gamma(\tfrac{1}{2}-s)\,\Gamma(\alpha+\beta+\gamma+s)\,\Gamma(\alpha+\beta'+\gamma+s)\,u^{-s}\,ds$$

$$=\frac{-1}{2\pi i}\int\Gamma(-2s)\,2^{2s+1}\sqrt{(\pi)}\,\Gamma(\alpha+\beta+\gamma+s)\,\Gamma(\alpha+\beta'+\gamma+s)\,u^{-s}\,ds,$$

the upper or lower sign being taken as x lies within or without the circle ABC. And the last expression may be written in the form

$$\left(\frac{-1}{2\pi i}\right)^2\int\sqrt{(\pi)}\,2^s\,\Gamma(\alpha+\beta+\gamma+\tfrac{1}{2}s)\,\Gamma(\alpha+\beta'+\gamma+\tfrac{1}{2}s)\,ds$$

$$\times\int\Gamma(\phi-s)\,\Gamma(-\phi)\,\{\sqrt{(1/u)}-1\}^{\phi}\,d\phi,$$

provided $|\arg\{\sqrt{(1/u)}-1\}|<\pi$.

Interchanging the order of integration, we have

$$\left(-\frac{1}{2\pi i}\right)^2\int\Gamma(-\phi)\,\{\sqrt{(1/u)}-1\}^{\phi}\,d\phi$$

$$\times\int 2^{\phi-1}\,\Gamma(\alpha+\beta+\gamma+\tfrac{1}{2}s)\,\Gamma(\alpha+\beta'+\gamma+\tfrac{1}{2}s)\,\Gamma(\tfrac{1}{2}\phi-\tfrac{1}{2}s)\,\Gamma\{\tfrac{1}{2}(\phi+1)-\tfrac{1}{2}s\}\,ds$$

$$=[\text{by H. F. § 16}]\,\frac{-1}{2\pi i}\int\Gamma(-\phi)\,\{\sqrt{(1/u)}-1\}^{\phi}\,2^{\phi}\,\Gamma(\tfrac{1}{2}\phi+\alpha+\beta+\gamma)$$

$$\times\frac{\Gamma\{\frac{1}{2}(\phi+1)+\alpha+\beta+\gamma\}\,\Gamma(\frac{1}{2}\phi+\alpha+\beta'+\gamma)\,\Gamma\{\frac{1}{2}(\phi+1)+\alpha+\beta'+\gamma\}}{\Gamma(\phi+\frac{1}{2}+2\alpha+2\gamma+\beta+\beta')}\,d\phi$$

$$=\frac{\pi\,\Gamma(2\alpha+2\beta+2\gamma)\,\Gamma(2\alpha+2\beta'+2\gamma)}{\Gamma(1-\alpha'+\alpha)\,2^{1\alpha+2\beta+1\gamma+2\beta'-2}}$$

$$\times F[2\alpha+2\beta+2\gamma,\ 2\alpha+2\beta'+2\gamma;\ 1+\alpha-\alpha';\ \tfrac{1}{2}\{1-\sqrt{(1/u)}\}].$$

We therefore have

$$P_1w^{-\alpha}u^{\gamma}e^{\pm\pi i\alpha}=\frac{\Gamma(\alpha+\beta'+\gamma)\,\Gamma(\alpha+\beta+\gamma)}{\Gamma(1+\alpha-\alpha')}$$

$$\times F[2\alpha+2\beta+2\gamma,\ 2\alpha+2\beta'+2\gamma;\ 1+\alpha-\alpha';\ \tfrac{1}{2}\{1-\sqrt{(1/u)}\}].$$

But by H. F. § 24, we have

$$P_1\begin{Bmatrix} 1\ ,\ \infty\ ,\ -1\ , \\ \alpha+\gamma,\ 2\beta,\ \alpha+\gamma,\ \sqrt{(1/u)} \\ \alpha'+\gamma,\ 2\beta',\ \alpha'+\gamma \end{Bmatrix}$$

$$= \left[\tfrac{1}{2}\{1+\sqrt{(1/u)}\}\right]^{\alpha+\gamma}\left[\tfrac{1}{2}\{1-\sqrt{(1/u)}\}\right]^{\alpha+\gamma} e^{\mp\pi\iota\alpha}$$

$$\times \frac{\Gamma(2\alpha+2\beta+2\gamma)\,\Gamma(2\alpha+2\beta'+2\gamma)}{\Gamma(1+\alpha-\alpha')}$$

$$\times F\left[2\alpha+2\beta+2\gamma,\ 2\alpha+2\beta'+2\gamma;\ 1+\alpha-\alpha';\ \tfrac{1}{2}\{1-\sqrt{(1/u)}\}\right],$$

the upper or lower sign being taken as $\sqrt{(1/u)}$ has its imaginary part negative or positive.

If then we take $\arg\sqrt{(1/u)}$ to be $-\frac{1}{2}\pi$, where x lies just within the arc BC, 0 on the arc ABC, $\frac{1}{2}\pi$ just outside the arc BC, $\sqrt{(1/u)}$ will have its imaginary part negative or positive as x lies within or without the circle ABC. And, since u is greater than 1 on the arc AC and less than 1 on the arc AB, we have

$$\left[\tfrac{1}{2}\{1+\sqrt{(1/u)}\}\right]^{\alpha+\gamma}\left[\tfrac{1}{2}\{1-\sqrt{(1/u)}\}\right]^{\alpha+\gamma} = \left[\tfrac{1}{4}\{1-(1/u)\}\right]^{\alpha+\gamma} = \frac{w^{\alpha+\gamma}}{2^{2\alpha+2\gamma}},$$

the final many-valued function having its stereotyped meaning.

Therefore

$$P_1\begin{Bmatrix} a,\ b,\ c \\ \alpha,\ \beta,\ \gamma\ ,\ x \\ \alpha',\ \beta',\ \gamma+\frac{1}{2} \end{Bmatrix} w^{-\alpha}u^{\gamma}e^{\pm\pi\iota\alpha}$$

$$= P_1\begin{Bmatrix} 1\ ,\ \infty\ ,\ -1 \\ \alpha+\gamma,\ 2\beta,\ \alpha+\gamma,\ \sqrt{(1/u)} \\ \alpha'+\gamma,\ 2\beta',\ \alpha'+\gamma \end{Bmatrix} w^{-(\alpha+\gamma)}$$

$$\times \frac{\pi\, 2^{1+2\alpha'+2\gamma} e^{\pm\pi\iota\alpha}}{\Gamma(\alpha+\beta+\gamma+\frac{1}{2})\,\Gamma(\alpha+\beta'+\gamma+\frac{1}{2})}.$$

But with the specifications which have been used throughout

$$u^{\gamma}v^{\gamma}w^{\gamma}=1 \quad [\text{H. F. § 22}].$$

We therefore have the given result.

The *form* of the relation may be readily verified by considering in Riemann's manner the singularities of the functions which occur on the two sides of the quantity.

§ 37. Suppose now that we put $a=1$, $b=\infty$, $c=0$, $\gamma=0$; then we have the complete form* of Riemann's transformation of the second degree previously quoted in § 3 :—

$$P_1\begin{Bmatrix}1, \infty, 0\\ \alpha, \beta, 0, x\\ \alpha', \beta', \frac{1}{2}\end{Bmatrix} = \frac{\pi 2^{1+2\alpha'}}{\Gamma(\alpha+\beta+\frac{1}{2})\Gamma(\alpha+\beta'+\frac{1}{2})} P_1\begin{Bmatrix}1, \infty, -1\\ \alpha, 2\beta, \alpha, \sqrt{x}\\ \alpha', 2\beta', \alpha'\end{Bmatrix},$$

wherein $\alpha+\alpha'+\beta+\beta'=\frac{1}{2}$.

As Riemann did not define the P-function except as to an arbitrary constant multiplier he did not give, and in fact could not in the nature of things obtain, the constant term in this relation.

From this result we may readily deduce the fundamental transformation of § 15. We have, when $\alpha+\alpha'+\beta+\beta'=\frac{1}{2}$, and $R(y)>0$,

$$F\{\alpha+\beta, \alpha+\beta';\ 1-\alpha'+\alpha;\ 1-y^2\} = F\{2\alpha+2\beta, 2\alpha+2\beta';\ 1-\alpha'+\alpha;\ \tfrac{1}{2}(1-y)\}.$$

For, if we express the two P-functions in terms of hypergeometric series, we get [H. F., § 24]

$$(x-1)^\alpha \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+\beta')}{\Gamma(1-\alpha'+\alpha)} F\{\alpha+\beta, \alpha+\beta';\ 1-\alpha'+\alpha;\ 1-x\}$$

$$= \frac{\pi 2^{1+2\alpha'}}{\Gamma(\alpha+\beta+\frac{1}{2})\Gamma(\alpha+\beta'+\frac{1}{2})} \{\tfrac{1}{2}(\sqrt{x}-1)\}^\alpha \{\tfrac{1}{2}(\sqrt{x}+1)\}^\alpha$$

$$\times \frac{\Gamma(2\alpha+2\beta)\Gamma(2\alpha+2\beta')}{\Gamma(1-\alpha'+\alpha)} F\{2\alpha+2\beta, 2\alpha+2\beta';\ 1-\alpha'+\alpha;\ \tfrac{1}{2}(1-\sqrt{x})\},$$

wherein $|\arg\sqrt{x}|<\frac{1}{2}\pi$. With the specifications used throughout, we have

$$(x-1)^\alpha = (\sqrt{x}-1)^\alpha (\sqrt{x}+1)^\alpha.$$

We therefore have

$$F\{\alpha+\beta, \alpha+\beta';\ 1-\alpha'+\alpha;\ 1-x\} = F\{2\alpha+2\beta, 2\alpha+2\beta';\ 1-\alpha'+\alpha;\ \tfrac{1}{2}(1-\sqrt{x})\}.$$

Putting $y=\sqrt{x}$, we have, when $R(y)>0$, the given relation.

In a similar manner we may deduce the transformation of § 24.

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* Cf. Riemann, *Oeuvres Mathématiques* (Paris, 1898), p. 73, formula (*A*).

PART II.

Asymptotic expansions of generalised Legendre functions.

§ 38. It will be evident to the reader who has followed the previous investigations that the comparative simplicity of the contour integrals for generalised Legendre functions, which have been introduced in the previous theory, may be expected to lead to simple proofs of many formulæ in the theory of such functions and to wide generalisations of known results. In the ensuing paragraphs we propose to deal with one of the most interesting and important branches of the theory, that of the asymptotic values of the functions. We shall investigate, in the first place, the asymptotic expansions of $P_n^m(x)$ and $Q_n^m(x)$ when $|n|$ is large. In the second place we shall consider the asymptotic expansions of the same functions when $|m|$ is large. All the variables x, m, and n are supposed in the general investigations to have general complex values. Particular expansions, when some of the variables are real, are deduced as subordinate results, or obtained by special methods.

When n is a positive integer, and $\cos\theta$ is real and positive, the aspmytotic approximation to $P_n(\cos\theta)$ was first determined by Laplace*. Laplace gave two investigations, neither satisfactory from the point of view of modern pure mathematics. Another investigation of the same limited problem is due to Bonnet†; criticisms by Heine of Bonnet's theory will be found in the latter's treatise‡. Todhunter§, in his text book, reproduces some of these investigations. Heine himself¶ investigated the same problem, when n is a positive integer and x is complex, for both the functions $P_n(x)$ and $Q_n(x)$. He did not, however, estimate the magnitude of the remainder after the terms taken in his approximation, and did not therefore show that the results were valid in Poincaré's‖ sense. Darboux**, in his famous *Mémoire sur l'approximation des fonctions de très-grands nombres*, appears to have been the first to obtain an expression for the magnitude of the remainder in the asymptotic formula for $P_n(x)$, when n is a large positive integer and x is either real and comprised between -1 and 1

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* Laplace, *Mécanique Celeste*, T. 5, Livre XI., §3; and Supplément au T. 5.
† Ossian Bonnet, *Liouville*, T. 17 (1852), pp. 265–300, especially pp. 270–277.
‡ Heine, *Handbuch der Kugelfunctionen* (1878), T. 1, pp. 171 and 179.
§ Todhunter, *Functions of Laplace, Lamé, and Bessel* (1875), Chapter 7.
¶ Heine, *loc. cit.*, pp. 171–182.
‖ Poincaré, *Acta Mathematica*, T. 8, pp. 295, et seq.
** Darboux, *Liouville* (1878), Ser. 3, T. 4, pp. 22 and 39.

or complex. Darboux also obtained (*loc. cit.*, p. 402) a similar formula for $Q_n(x)$, when n is a large positive integer, and extended his results to the well-known polynomials of Jacobi and Tchebychef. But his method only applied to cases in which n is a large positive integer. Stieltjes* attacked the problem of the asymptotic value of $P_n(x)$, when n is a large positive integer and x is real and comprised between -1 and 1, by the theory of contour integrals, and obtained a series convergent for a certain range of values of the variable and rigorously asymptotic outside this range. Hobson†, relying on a formula of Darboux‡, obtained expansions for $P_n^m(\cos\theta)$, $Q_n^m(\cos\theta)$, when $\cos\theta$ is positive, m and n are real, and n large and positive but not necessarily integral. He also obtained asymptotic formulæ, when x is real and greater than unity, for $P_n^m(x)$ and $Q_n^m(x)$ under similar limitations for m and n. And by estimating the remainders in the earlier set of these formulæ he established them in Poincaré's sense. Finally, reference must be made to another important contribution of Stieltjes§ who, by his proof that the asymptotic expansion of $\Gamma(x)$ is valid for general complex values of x when $|\arg x| < \pi$, made possible the subjoined theory of such expansions.

In the present investigation I show by means of the asymptotic expansion for $\Gamma(s)$ that an asymptotic expansion can be obtained for $Q_n^m(x)$, when m, n, and x have general complex values and $|n|$ is large, provided $|\arg n| < \pi - \epsilon$, where ϵ is a finite positive quantity, however small, independent of n. It follows that a similar expansion can be obtained for $P_n^m(x)$, when $|n|$ is large, provided $|\arg(\pm n)| < \pi - \epsilon$, that is to say, when n is not near the positive or negative directions of the real axis. The special cases, when x is real and n is real and positive, are considered in detail. The generalisation thus made is not merely of great theoretical interest. Such formulæ would be useful in the consideration of expansions in terms of Mehler's¶ conal harmonics.

Finally, at the close of the present Part of this memoir, I investigate the asymptotic expansion of $P_n^m(x)$ and $Q_n^m(x)$, when m, n, and x have general complex values and $|m|$ is

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* Stieltjes, *Annales de la Faculté des Sciences de Toulouse* (1890), T. 4, pp. G 1–17.

† Hobson, *Phil. Trans., Royal Society*, Vol. CLXXXVII. (1896), A. pp. 486–491.

‡ Darboux, *Liouville*, Séries 3, T. 2, pp. 291–312.

§ Stieltjes, *Liouville* (1889), Séries 4, T. 5, pp. 425–444.

¶ Mehler, *Crelle*, Vol. LXVIII., pp. 134–150; also Hobson, *loc. cit.*, p. 529, and *Transactions of the Cambridge Philosophical Society* (1887), Vol. XIV., pp. 211–236.

large. I am not aware that any such investigations have been previously given, though they are necessary to establish the convergence of many expansions in terms of generalised Legendre functions.

§ 39. *The asymptotic expansion $Q_n^m(x)$ when n, m, and x have general complex values, $|n|$ is large, and $|\arg n| < \pi - \epsilon$, where ϵ is a finite positive quantity, however small, independent of n, $|m|$ and $|x|$ being finite.*

Let $S_r(x)$ be the r^{th} simple Bernoullian function of x, which is the only algebraic solution of the difference equation

$$f(x+1) - f(x) = x^r,$$

which vanishes with x. $S_r(x)$ is thus a polynomial in x of degree $(r+1)$.

Let the quantities K_r^m be determined when $r < N$ by the expansion

$$\exp\left[\sum_{r=1}^{N} \frac{(-)^{r-1}}{r x^r}\{S_r(m+1) - S_r(\tfrac{3}{2})\}\right] = \sum_{r=0}^{\infty} \frac{K_r^m}{x^r}.$$

Let the functions $\Phi_q^m(\xi)$ be given by the equality

$$\Phi_q^m(\xi) = \sum_{t=0}^{\infty} \frac{\Gamma(m+t+\frac{1}{2})\,\xi^{2t}}{\Gamma(t+1)} \sum_{p=0}^{q} \frac{(-)^p K_{q-p}^m t^p}{p!\,\Gamma(q-p+\frac{1}{2}-m)},$$

wherein $\xi = x - \sqrt{(x^2-1)}$ and $|\xi| < 1 - \epsilon_1$, where ϵ_1 is a finite positive quantity, however small, independent of n.

Then, if $|n|$ be large and $|\arg n| < \pi - \epsilon$, where ϵ is a finite positive quantity, however small, independent of n, we have the asymptotic equality

$$Q_n^m(x) = \frac{\sin\pi(n+m)}{\sin\pi n} \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m)} (1-\xi^2)^{\frac{m}{2}} \xi^{n+1} n^{m-\frac{1}{2}} \times \left\{\sum_{q=0}^{N} \frac{\Gamma(q+\frac{1}{2}-m)}{n^q} \Phi_q^m(\xi) + \frac{J_N}{n^N}\right\},$$

where, for any assigned finite value of N, $|J_N|$ can be made as small as we please by taking $|n|$ greater than a sufficiently large positive quantity. In this case we shall sometimes say that $|J_N|$ tends uniformly to zero as $|n|$ tends to infinity.

We have seen that [§ 25]

$$Q_n^m(x)\,\frac{\Gamma(m+\frac{1}{2})\sin\pi n}{\Gamma(\frac{1}{2})\sin\pi(n+m)}\,(1-\xi^2)^{-m}\,\xi^{-n-1}$$

$$=-\frac{1}{2\pi\iota}\int\Gamma(m+\tfrac{1}{2}+s)\,\Gamma(-s)\,\frac{\Gamma(n+s+m+1)}{\Gamma(n+s+\frac{3}{2})}\,(-\xi^2)^s\,ds,$$

wherein $|\arg\xi|<\pi$, $|\arg(-\xi^2)|<\pi$.

If $|\xi|<1$ we may take the integral along a contour closely embracing the positive half of the real axis. If, then, $|\arg n|<\pi-\epsilon$, where ϵ is a finite positive quantity independent of n, we shall have, when $|n|$ is sufficiently large, $|s+n|$ as large as we please at all points on the contour.

Therefore by the asymptotic expansion of the gamma-function

$$\frac{\Gamma(n+s+m+1)}{\Gamma(n+s+\frac{3}{2})}$$

$$=\frac{1}{(n+s)^{\frac{1}{2}-m}}\exp\left\{\sum_{r=1}^{N}\frac{(-)^{r-1}}{r\,(n+s)^r}\left[S_r(m+1)-S_r(\tfrac{3}{2})\right]+\frac{{}_1J_N}{(n+s)^N}\right\},$$

where $|{}_1J_N|$ can be made as small as we please for any assigned finite value of N, however large, by taking $|n|$ to have any value greater than a sufficiently large positive quantity.

Thus

$$\frac{\Gamma(n+s+m+1)}{\Gamma(n+s+\frac{3}{2})}=\sum_{r=0}^{N}\frac{K_r^m}{(n+s)^{r+\frac{1}{2}-m}}+\frac{{}_2J_N(s+n)}{(n+s)^{N+\frac{1}{2}-m}},$$

where $|{}_2J_N(s+n)|$ possesses a similar property.

The previous integral is therefore equal to

$$-\frac{1}{2\pi\iota}\sum_{r=0}^{N}K_r^m\int\frac{\Gamma(-s)\,\Gamma(m+s+\frac{1}{2})}{(n+s)^{r+\frac{1}{2}-m}}\,(-\xi^2)^s\,ds+\frac{{}_3J_N}{n^{N+\frac{1}{2}-m}},$$

where

$${}_3J_N=-\frac{1}{2\pi\iota}\int\frac{{}_2J_N(s+n)\,\Gamma(-s)\,\Gamma(m+s+\frac{1}{2})}{(1+s/n)^{N+\frac{1}{2}-m}}\,(-\xi^2)^s\,ds$$

$$=\sum_{t=0}^{\infty}\frac{{}_3J_N(t+n)\,\Gamma(m+t+\frac{1}{2})}{(1+t/n)^{N+\frac{1}{2}-m}\,\Gamma(t+1)}\,\xi^{2t}.$$

Hence $$|{}_3J_N|<\eta\sum_{t=0}^{\infty}\frac{|\Gamma(m+t+\frac{1}{2})|}{\Gamma(t+1)}\,|\xi^{2t}|,$$

where, for any assigned value of N, the positive quantity η can be made as small as we please by taking $|n|$ greater than a sufficiently large positive quantity. But the last series is convergent when $|\xi|<1$. Hence $|{}_3J_N|$ can be made as small as we please by taking $|n|$ sufficiently large, provided $|\xi|<1-\epsilon_1$, where ϵ_1 is a finite positive quantity, however small, independent of n.

Consider next the integral

$$-\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\,\Gamma(m+s+\frac{1}{2})}{(1+s/n)^{r+\frac{1}{2}-m}}(-\xi^2)^s\,ds.$$

It is equal to

$$\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)\,(1+t/n)^{r+\frac{1}{2}-m}}\xi^{2t}$$

$$=\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)}\xi^{2t}\left(-\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(\phi+r+\frac{1}{2}-m)\,\Gamma(-\phi)}{\Gamma(r+\frac{1}{2}-m)}\frac{t^\phi}{n^\phi}\,d\phi,$$

since $|\arg(t/n)|<\pi$, the integral being taken along a contour parallel to the imaginary axis leaving the origin on the right and the point $m-r-\frac{1}{2}$ on the left. And the last series may be written

$$\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)}\xi^{2t}\sum_{p=0}^{N}\frac{\Gamma(p+r+\frac{1}{2}-m)}{\Gamma(p+1)\,\Gamma(r+\frac{1}{2}-m)}\frac{(-)^p t^p}{n^p}+\frac{{}_0I_N}{n^N},$$

where

$${}_0I_N=-\frac{1}{2\pi\iota}\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)}\xi^{2t}\int\frac{\Gamma(\phi+N+r+\frac{1}{2}-m)\,\Gamma(-\phi-N)}{n^\phi\Gamma(r+\frac{1}{2}-m)}\,t^{\phi+N}d\phi,$$

the last integral being taken along a contour parallel to the imaginary axis, cutting the real axis between 0 and 1, with a loop, if necessary, to ensure that $m-N-r-\frac{1}{2}$ is to the left of the contour. Thus $|{}_0I_N|$ can be made as small as we please for any assigned value of N by sufficiently increasing $|n|$.

We therefore have

$$Q_n^m(x)\frac{\Gamma(m+\frac{1}{2})\sin\pi n}{\Gamma(\frac{1}{2})\sin\pi(n+m)}(1-\xi^2)^{-m}\xi^{-n-1}$$

$$=\sum_{r=0}^{N}\frac{K_r^m}{n^{r+\frac{1}{2}-m}}\sum_{p=0}^{N}\frac{(-)^p}{\Gamma(p+1)\,n^p}\frac{\Gamma(p+r+\frac{1}{2}-m)}{\Gamma(r+\frac{1}{2}-m)}\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)}\xi^{2t}t^p$$

$$+\frac{{}_1I_N}{n^{N+\frac{1}{2}-m}},$$

where $|{}_1I_N|$ possesses the same property as $|{}_0I_N|$,

$$=n^{m-\frac{1}{2}}\left\{\sum_{q=0}^{N}\frac{\Gamma(q+\frac{1}{2}-m)}{n^q}\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)}\xi^{2t}\sum_{p=0}^{q}\frac{(-)^p t^p K^m{}_{q-p}}{p!\,\Gamma(q-p+\frac{1}{2}-m)}+\frac{I_N}{n^N}\right\}.$$

We thus have the given result.

§ 40. *We will now show that the evaluation of the first two terms of the previous expansion leads to the asymptotic equality*

$$Q_n^m(x)=\sqrt{\frac{\pi}{n}}\,\frac{\sin\pi(n+m)}{\sin\pi n}\,\xi^{n+1}(1-\xi^2)^{-\frac{1}{4}}n^m\left\{1+\frac{{}_mC_1}{n}+\ldots\right\},$$

where $${}_mC_1=(m-\tfrac{1}{2})\left\{\tfrac{1}{2}(m+\tfrac{3}{2})+(m+\tfrac{1}{2})\frac{\xi^2}{1-\xi^2}\right\}.$$

In this result $|n|$ is large, $|\arg n|<\pi-\epsilon$, where ϵ is positive and independent of n, $|\xi|<1-\epsilon_1$, and $|\arg\xi|<\pi$. And the equality is truly asymptotic in Poincaré's sense.

We know that

$$S_1(a)=\tfrac{1}{2}(a^2-a),$$
$$S_2(a)=\tfrac{1}{3}a^3-\tfrac{1}{2}a^2+\tfrac{1}{6}a.$$
$$\&c.$$

Hence

$$S_1(m+1)-S_1(\tfrac{3}{2})=\tfrac{1}{2}(m^2+m-\tfrac{3}{4})=\tfrac{1}{2}(m+\tfrac{3}{2})(m-\tfrac{1}{2}).$$

Also $$\exp\left\{\frac{m^2+m-\frac{3}{4}}{2x}\right\}=K_0^m+\frac{K_1^m}{x}+\frac{(\cdot\cdot)}{x^2}+\ldots.$$

Therefore $$K_0^m=1,\quad K_1^m=\tfrac{1}{2}(m+\tfrac{3}{2})(m-\tfrac{1}{2}).$$

Hence

$$\Phi_0^m(\xi)=\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)\,\Gamma(\frac{1}{2}-m)}\xi^{2t}=\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2}-m)}(1-\xi^2)^{-m-\frac{1}{2}},$$

and

$$\Phi_1^m(\xi)=\sum_{t=0}^{\infty}\frac{\Gamma(m+t+\frac{1}{2})}{\Gamma(t+1)}\xi^{2t}\left\{\frac{(m+\frac{3}{2})(m-\frac{1}{2})}{2\Gamma(\frac{3}{2}-m)}-\frac{t}{\Gamma(\frac{1}{2}-m)}\right\}$$

$$=-\tfrac{1}{2}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2}-m)}(m+\tfrac{3}{2})(1-\xi^2)^{-m-\frac{1}{2}}-\xi^2\frac{\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2}-m)}(1-\xi^2)^{-m-\frac{3}{2}}.$$

Thus, as regards the first two terms, the previous expansion may be written

$$Q_n^m(x) = \frac{\sin\pi(n+m)}{\sin\pi n}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m)}\,\xi^{n+1}n^{m-\frac{1}{2}}\left\{\frac{\Gamma(m+\frac{1}{2})}{(1-\xi^2)^{\frac{1}{4}}}\right.$$

$$+\frac{\Gamma(\frac{3}{2}-m)}{n}\left[-\tfrac{1}{2}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2}-m)}(m+\tfrac{3}{2})(1-\xi^2)^{-\frac{1}{4}}\right.$$

$$\left.\left.-\xi^2\frac{\Gamma(m+\frac{3}{2})}{\Gamma(\frac{1}{2}-m)}(1-\xi^2)^{-\frac{3}{4}}\right]+\ldots\right\}.$$

We thus have the given result.

Hobson's formula [(66), p. 491], apart from an obvious misprint, is the particular case of this formula when n is real and positive and $x>1$.

§ 41. *The asymptotic expansion of* $Q_n^m(x) = G_n^m(\xi)$, *when* $|\xi|=1$, *and* $|n|$ *is large,* $|m|$ *being finite.*

When $|\xi|=1$, $Q_n^m(x)$ lies on the cross cut from $x=-1$ to $x=1$. We shall therefore work with the function $G_n^m(\xi)$, which, as previously defined in § 32, has no cross cut when $|\xi|=1$ ($\xi\neq 1$ or -1).

Put $\xi=e^{i\theta}$, where θ is real and $0<|\theta|<\pi$, and choose $(-\xi^2)$ so that $|\arg(-\xi^2)|<\pi$. Then we have

$$G_n^m(\xi)\,\frac{\Gamma(m+\frac{1}{2})\sin\pi n}{\Gamma(\frac{1}{2})\sin\pi(m+n)}\,(1-\xi^2)^{-m}\,\xi^{-n-1}$$

$$=-\frac{1}{2\pi i}\int\Gamma(m+\tfrac{1}{2}+s)\,\Gamma(-s)\,\frac{\Gamma(n+s+m+1)}{\Gamma(n+s+\frac{3}{2})}\,(-\xi^2)^s\,ds.$$

On the left-hand side $|\arg(1-\xi^2)|<\pi$ and $|\arg\xi|<\pi$. On the right-hand side we take the contour of the integral to be parallel to the imaginary axis (with, of course, suitable loops, if necessary).

The integral will be convergent when $|\xi|=1$, provided $|\arg(-\xi^2)|<\pi$, which has been ensured by the conditions imposed upon θ.

Now, provided $|\arg n|<\frac{1}{2}\pi-\epsilon$, where ϵ is a finite positive quantity, however small, independent of n, we may repeat our former analysis of § 39, working with contours parallel to the imaginary axis instead of with contours closely embracing the

positive half of the real axis. And since with such new contours the equation

$$\Gamma(-s)(1+x)^s = -\frac{1}{2\pi\iota}\int\Gamma(-\phi)\Gamma(\phi-s)x^\phi d\phi$$

is valid* for all values of s and $|x|$, provided $|\arg x|<\pi$, $|\arg(1+x)|<\pi$, we shall arrive at the same asymptotic equality as before, provided $|\arg(-\xi^2)|<\pi-\varepsilon_2$, where ε_2 is a finite positive quantity, however small, independent of n.

Thus the asymptotic expansion previously obtained holds for $Q_n^m(x)=G_n^m(\xi)$, *when* $|\xi|=1$, *provided* $\epsilon_2<|\arg\xi|<\pi-\varepsilon_2$ *and* $|\arg n|<\frac{1}{2}\pi-\epsilon$, *and provided* $|\arg(1-\xi^2)|<\pi$. *Thus* $(1-\xi^2)^m=(1-\xi)^m(1+\xi)^m$, *where* $|\arg(1\pm\xi)|<\pi$.

§ 42. *We may now show that when* θ *is positive and* $\epsilon_2<\theta<\pi-\epsilon_2$, *when* $|n|$ *is large and* $|\arg n|<\frac{1}{2}\pi-\epsilon$, *we have the asymptotic equality*

$$Q_n^m(\cos\theta)=\sqrt{\left(\frac{\pi}{2n\sin\theta}\right)}\,n^m\frac{\sin\pi(n+m)}{\sin\pi n}$$
$$\times\left[\cos\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}\left\{1+\frac{(m-\frac{1}{2})(m+\frac{3}{2})}{2n}\right\}\right.$$
$$\left.-\frac{m^2-\frac{1}{4}}{2n\sin\theta}\sin\{(n+\tfrac{3}{2})\theta+\tfrac{1}{4}\pi\}+\frac{(\cdot\cdot)}{n^2}+\ldots\right].$$

We have seen in § 33 that

$$Q_n^m(\cos\theta)=\tfrac{1}{2}\{G_n^m(e^{\iota\theta})+G_n^m(e^{-\iota\theta})\}.$$

We therefore have, by the result of the two previous paragraphs, under the assigned conditions for θ and n,

$$Q_n^m(\cos\theta)=\sqrt{(\pi/n)}\,\frac{\sin\pi(n+m)}{2\sin\pi n}$$
$$\times n^m\left[e^{(n+1)\iota\theta}(1-e^{2\iota\theta})^{-\frac{1}{2}}\left\{1+\frac{{}_mC_1(\theta)}{n}+\frac{(\cdot\cdot)}{n^2}+\ldots\right\}\right.$$
$$\left.+e^{-(n+1)\iota\theta}(1-e^{-2\iota\theta})^{-\frac{1}{2}}\left\{1+\frac{{}_mC_1(-\theta)}{n}+\frac{(\cdot\cdot)}{n^2}+\ldots\right\}\right],$$

where $${}_mC_1(\theta)=(m-\tfrac{1}{2})\left\{\tfrac{1}{2}(m+\tfrac{3}{2})+(m+\tfrac{1}{2})\frac{e^{2\iota\theta}}{1-e^{2\iota\theta}}\right\}.$$

* See the author's paper, *Quarterly Journal of Mathematics*, Vol. XXXVIII., pp. 108–116.

But since θ is positive

$$(1-e^{2\iota\theta})^{-\frac{1}{2}} = (2\sin\theta)^{-\frac{1}{2}} e^{-\frac{1}{2}\iota\theta+\frac{1}{4}\iota\pi},$$

$$(1-e^{-2\iota\theta})^{-\frac{1}{2}} = (2\sin\theta)^{-\frac{1}{2}} e^{\frac{1}{2}\iota\theta-\frac{1}{4}\iota\pi}.$$

Hence

$$Q_n^m(\cos\theta) = \sqrt{\left(\frac{\pi}{2n\sin\theta}\right)} \frac{\sin\pi(m+n)}{2\sin\pi n}$$

$$\times n^m \left\{ e^{(n+\frac{1}{2})\iota\theta+\frac{1}{4}\pi\iota}\left(1+\frac{{}_mC_1(\theta)}{n}+\ldots\right)+e^{-(n+\frac{1}{2})\iota\theta-\frac{1}{4}\pi\iota}\left(1+\frac{{}_mC_1(-\theta)}{n}+\ldots\right)\right\}.$$

Now

$${}_mC_1(\theta) = (m-\tfrac{1}{2})\left\{\tfrac{1}{2}(m+\tfrac{3}{2})+\iota(m+\tfrac{1}{2})\frac{e^{\iota\theta}}{2\sin\theta}\right\}.$$

Hence

$$Q_n^m(\cos\theta) = \sqrt{\left(\frac{\pi}{2n\sin\theta}\right)} n^m \frac{\sin\pi(m+n)}{\sin\pi n}$$

$$\times\left[\cos\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}\left\{1+\frac{(m-\frac{1}{2})(m+\frac{3}{2})}{2n}\right\}\right.$$

$$\left.-\frac{m^2-\frac{1}{4}}{2n\sin\theta}\sin\{(n+\tfrac{3}{2})\theta+\tfrac{1}{4}\pi\}+\frac{(\cdot\cdot)}{n^2}+\ldots\right].$$

We thus have the given result.

When $m=0$ we have the asymptotic expansion, valid when $|n|$ is large, $|\arg n| < \frac{1}{2}\pi-\epsilon$ and $\epsilon_2 < \theta < \pi-\epsilon_1$,

$$Q_n(\cos\theta) = \sqrt{\left(\frac{\pi}{2n\sin\theta}\right)}\left[\cos\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}\left(1-\frac{3}{8n}\right)\right.$$

$$\left.+\frac{\sin\{(n+\frac{3}{2})\theta+\frac{1}{4}\pi\}}{8n\sin\theta}+\frac{(\cdot\cdot)}{n^2}+\ldots\right].$$

This should agree with the result of Heine and Hobson for the case when n and m are real. In each case there is a slight divergence. Heine (*loc. cit.*, p. 182) obtains for the

coefficient of $\frac{1}{4}\left(\frac{\pi}{2n^3\sin\theta}\right)^{\frac{1}{2}}$, the terms

$$\sin\tfrac{1}{4}\pi\sin(n+\tfrac{1}{2})\theta+\tfrac{1}{2}\cot\theta\cos\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\},$$

Hobson (*loc. cit.*, p. 490) gives the terms

$$-\cos\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}+\tfrac{1}{2}\cot\theta\sin\{(n+\tfrac{1}{2})\theta\}.$$

It appears that the two terms are

$$-\cos\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}+\tfrac{1}{2}\cot\theta\sin\{(n+\tfrac{1}{2})\theta+\tfrac{1}{4}\pi\}.$$

§ 43. *We may next show that when θ is positive and $\epsilon_2<\theta<\pi-\epsilon_2$, when $|n|$ is large and $|\arg n|<\frac{1}{2}\pi-\epsilon$, we have the asymptotic equality*

$$Q_n^m(-\cos\theta)=\sqrt{\left(\frac{\pi}{2n\sin\theta}\right)}\frac{\sin\pi(n+m)}{\sin\pi n}$$
$$\times n^m\left[\cos\{(n+\tfrac{1}{2})(\theta-\pi)-\tfrac{1}{4}\pi\}+\frac{(\cdot\cdot)}{n}+\ldots\right].$$

As has been seen in § 34, we have, under the assigned conditions for θ,

$$Q_n^m(-\cos\theta)=\tfrac{1}{2}\{G_n^m(e^{\iota\theta})\,e^{-(n+1)\pi\iota}+G_n^{m_1}(e^{-\iota\theta})\,e^{(n+1)\pi\iota}\}$$
$$=\sqrt{(\pi/n)}\frac{\sin\pi(n+m)}{2\sin\pi n}n^m\left[e^{(n+1)\iota(\theta-\pi)}(1-e^{2\iota\theta})^{-\frac{1}{2}}\left\{1+\frac{{}^mC_1(\theta)}{n}+\ldots\right\}\right.$$
$$\left.+e^{-(n+1)\iota(\theta-\pi)}(1-e^{-2\iota\theta})^{-\frac{1}{2}}\left\{1+\frac{{}^mC_1(-\theta)}{n}+\ldots\right\}\right],$$

which yields the result stated: it is but another way of writing the result of § 42, and verifies the range there given for $|\arg\xi|$.

§ 44. *The asymptotic expansion for $P_n^m(x)$, when $|n|$ is large, $|m|$ and $|x|$ are finite, $|\xi|<1-\epsilon_1$, and $|arg(\pm n)|<\pi-\epsilon$, where ϵ and ϵ_1 are positive quantities independent of n.*

We have, in § 12, obtained the formula

$$P_n^m(x)=\frac{\tan n\pi}{\pi}\{Q_n^m(x)-Q^m{}_{-n-1}(x)\}.$$

Also we have obtained the asymptotic equality

$$Q_n^m(x) = \frac{\sin\pi(n+m)}{\sin\pi n}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m)}\,(1-\xi^2)^m\,\xi^{n+1}\,n^{m-\frac{1}{2}} \times \left\{\sum_{q=0}^{N}\frac{\Gamma(q+\frac{1}{2}-m)}{n^q}\,\Phi_q^m(\xi)+\frac{J_N}{n^N}\right\},$$

where $|n|$ is large and $|\arg n|<\pi-\varepsilon$.

If, then, $|\arg(\pm n)|<\pi-\epsilon$ and $|n|$ is large, $|\xi|<1-\varepsilon_1$, we have

$$P_n^m(x) = \frac{\sin\pi(n+m)}{\pi\cos n\pi}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m)}\,(1-\xi^2)^m\,\xi^{n+1}\,n^{m-\frac{1}{2}} \times \left\{\sum_{q=0}^{N}\frac{\Gamma(q+\frac{1}{2}-m)}{n^q}\,\Phi_q^m(\xi)+\frac{{}_1J_N}{n^N}\right\}$$

$$+\frac{\sin\pi(m-n)}{\pi\cos n\pi}\,\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+m)}\,(1-\xi^2)^m\,\xi^{-n}\,(-n-1)^{m-\frac{1}{2}} \times \left\{\sum_{q=0}^{N}\frac{\Gamma(q+\frac{1}{2}-m)}{(-n-1)^q}\,\Phi_q^m(\xi)+\frac{{}_2J_N}{n^N}\right\},$$

where $|{}_1J_N|$ and $|{}_2J_N|$ can be made as small as we please, for any assigned value of N, by taking $|n|$ greater than a sufficiently large positive quantity.

If in this result we confine ourselves to the initial terms, we have

$$P_n^m(x) = \frac{\sin\pi(n+m)}{\sqrt{(\pi)}\cos\pi n}\,(1-\xi^2)^{-\frac{1}{2}}\,\xi^{n+1}\,n^{m-\frac{1}{2}}\left[1+\frac{{}_mC_1}{n}+\ldots\right]$$

$$+\frac{\sin\pi(m-n)}{\sqrt{(\pi)}\cos\pi n}\,(1-\xi^2)^{-\frac{1}{2}}\,\xi^{-n}\,(-n-1)^{m-\frac{1}{2}}\left[1-\frac{{}_mC_1}{n+1}+\ldots\right],$$

where

$${}_mC_1 = (m-\tfrac{1}{2})\left\{\tfrac{1}{2}(m+\tfrac{3}{2})+(m+\tfrac{1}{2})\,\frac{\xi^2}{1-\xi^2}\right\}.$$

We see that this perfectly general formula is not valid when n is near the positive or negative directions of the real axis.

§ 45. *We proceed now to consider in detail the dominant term furnished by the previous asymptotic expansion for* $P_n^m(x)$ *in the various cases which may arise.*

Put $\xi = (1/r)e^{i\theta}$ so that $|r|>1$ and $|\theta|<\pi$; and, further, put $n = u+iv$ where u and v are real. The terms of the two series which enter into the asymptotic expansion for $P_n^m(x)$ are respectively dominated by ξ^{n+1} and ξ^{-n}, and

$$|\xi^n| = \exp\{-\theta v - u\log r\}.$$

We therefore have the following cases.

Case I. (*a*):—u and v both positive; $I(\xi)$ positive or, if $I(\xi)$ is negative, $|\theta|<(u/v)\log r$.

In this case ξ^n is, when $|n|$ is large, negligible in comparison with ξ^{-n}.

Hence we have the asymptotic formula

$$P_n^m(x) = \frac{\sin\pi(m-n)}{\sqrt{(\pi)}\cos\pi n}(1-\xi^2)^{-\frac{1}{4}}\xi^{-n}(-n-1)^{m-\frac{1}{2}}\left\{1-\frac{{}_mC_1}{n+1}+\ldots\right\}.$$

But
$$\frac{\sin\pi(m-n)}{\cos\pi n} = \frac{e^{\pi\iota m}}{\iota},$$

if the terms in n of the lower exponential order be neglected.

Thus, since $(-n)^{m-\frac{1}{2}} = n^{m-\frac{1}{2}}e^{-\pi\iota(m-\frac{1}{2})}$,

$$P_n^m(x) = \frac{\xi^{-n}n^m}{\sqrt{(\pi n)}(1-\xi^2)^{\frac{1}{4}}}\left\{1+\frac{(\cdot\cdot)}{n}+\ldots\right\}.$$

Case I. (*b*):—u and v both positive; $I(\xi)$ negative and $|\theta|>(u/v)\log r$.

In this case ξ^{-n} is negligible in comparison with ξ^n, and we have

$$P_n^m(x) = \frac{\sin\pi(m+n)}{\sqrt{(\pi)}\cos\pi n}(1-\xi^2)^{-\frac{1}{4}}\xi^{n+1}n^{m-\frac{1}{2}}\left\{1+\frac{{}_mC_1}{n}+\ldots\right\}$$

But
$$\frac{\sin\pi(m+n)}{\cos\pi n} = -\frac{e^{-\pi\iota m}}{\iota},$$

terms of lower exponential order in n being neglected.

Therefore

$$P_n^m(x) = \frac{\iota e^{-\pi\iota m}\xi^{n+1}n^m}{\sqrt{(\pi n)}(1-\xi^2)^{\frac{1}{4}}}\left\{1+\frac{(\cdot\cdot)}{n}+\ldots\right\},$$

Case II. (*a*):—u positive, v negative; $I(\xi)$ negative or, if $I(\xi)$ is positive, $\theta<(u/|v|)\log r$.

In this case ξ^n is negligible in comparison with ξ^{-n}, and we have

$$P_n^m(x) = \frac{\xi^{-n} n^m}{\sqrt{(\pi n)}\,(1-\xi^2)^{\frac{1}{4}}} \left\{1 + \frac{(\cdot\cdot)}{n} + \ldots\right\}.$$

The form is thus the same as in Case I. (*a*).

Case II. (*b*):—u positive, v negative; $I(\xi)$ positive and $\theta > (u/|v|)\log r$. We have

$$P_n^m(x) = -\frac{\iota e^{\pi\iota m}\,\xi^{n+1}\,n^m}{\sqrt{(\pi n)}\,(1-\xi^2)^{\frac{1}{4}}} \left\{1 + \frac{(\cdot\cdot)}{n} + \ldots\right\}.$$

Cases III. and IV. when u is negative, can be derived from the preceding by the formula

$$P_n^m(x) = P^m{}_{-n-1}(x).$$

Throughout the previous investigations we have been limited by the condition $|\arg(\pm n)| < \pi - \varepsilon$. When $R(n)$ is positive and $|\theta v| < u\log r$, we have obtained the asymptotic formula

$$P_n^m(x) = \frac{\xi^{-n} n^m}{\sqrt{(\pi n)}\,(1-\xi^2)^{\frac{1}{4}}} \left\{1 + \frac{(\cdot\cdot)}{n} + \ldots\right\},$$

and the part of $P_n^m(x)$, which is discontinuous, when n passes over the real axis, is masked by the dominant term just written down. The expansion ceases to hold for general values of n, when n lies on the real axis, owing to the fact that the expression for $P_n^m(x)$ in § 29 is no longer valid for general complex values of m, when n is half an odd integer. Hence, when n lies on the positive half of the real axis, special investigations become necessary. The most important of these is given in § 48, *infra*.

§ 46. *The asymptotic expansion of* $P_n^m(\cos\theta)$, *when* $\varepsilon_2 < \theta < \pi - \varepsilon_2$, *where* ε_2 *is a finite positive quantity, however small, independent of* n.

We now proceed to show that, if $\varepsilon_2 < \theta < \pi - \varepsilon_2$, and $|\arg n| < \frac{1}{2}\pi - \varepsilon$, we have the asymptotic equality

$$P_n^m(\cos\theta) = \sqrt{\left(\frac{2}{\pi n \sin\theta}\right)} n^m \Big[\sin\{(n+\tfrac{1}{2})\theta + \tfrac{1}{2}m\pi + \tfrac{1}{4}\pi\} \times \left\{1 + \frac{(m-\frac{1}{2})(m+\frac{3}{2})}{2n}\right\} + \frac{m^2-\frac{1}{4}}{2\sin\theta}\,\frac{\cos\{(n+\frac{3}{2})\theta + \frac{1}{2}m\pi + \frac{1}{4}\pi\}}{n} + \frac{(\cdot\cdot)}{n^2} + \ldots\Big].$$

We have seen in § 33 that, under the assigned conditions,

$$\frac{-\iota\pi\sin(m+n)\pi}{\sin n\pi}P_n^m(\cos\theta)=G_n^m(e^{-\iota\theta})e^{-\frac{1}{2}m\pi\iota}-G_n^m(e^{\iota\theta})e^{\frac{1}{2}m\pi\iota}.$$

Also, by § 41,

$$G_n^m(e^{\iota\theta})=\sqrt{(\pi/n)}\frac{\sin\pi(n+m)}{\sin\pi n}e^{(n+1)\iota\theta}(1-e^{2\iota\theta})^{-\frac{1}{2}}n^m\left\{1+\frac{{}_mC_1(\theta)}{n}+\ldots\right\},$$

where

$${}_mC_1(\theta)=(m-\tfrac{1}{2})\left\{\tfrac{1}{2}(m+\tfrac{3}{2})+(m+\tfrac{1}{2})\frac{e^{2\iota\theta}}{1-e^{2\iota\theta}}\right\}.$$

Hence

$$\begin{aligned}-\epsilon\sqrt{(\pi n)}\,P_n^m(\cos\theta)\,n^{-m}&\\ &=e^{-(n+1)\iota\theta-\frac{1}{2}m\pi\iota}(1-e^{-2\iota\theta})^{-\frac{1}{2}}\left\{1+\frac{{}_mC_1(-\theta)}{n}+\ldots\right\}\\ &\quad-e^{(n+1)\iota\theta+\frac{1}{2}m\pi\iota}(1-e^{2\iota\theta})^{-\frac{1}{2}}\left\{1+\frac{{}_mC_1(\theta)}{n}+\ldots\right\}\\ &=e^{-(n+\frac{1}{2})\iota\theta-\frac{1}{2}m\pi\iota-\frac{1}{4}\pi\iota}(2\sin\theta)^{-\frac{1}{2}}\left\{1+\frac{{}_mC_1(-\theta)}{n}+\ldots\right\}\\ &\quad-e^{(n+\frac{1}{2})\iota\theta+\frac{1}{2}m\pi\iota+\frac{1}{4}\pi\iota}(2\sin\theta)^{-\frac{1}{2}}\left\{1+\frac{{}_mC_1(\theta)}{n}+\ldots\right\}.\end{aligned}$$

Thus

$$\begin{aligned}P_n^m(\cos\theta)=\sqrt{\left(\frac{2}{\pi n\sin\theta}\right)}n^m\Bigg[&\sin\{(n+\tfrac{1}{2})\theta+\tfrac{1}{2}m\pi+\tfrac{1}{4}\pi\}\\ &\times\left(1+\frac{(m-\frac{1}{2})(m+\frac{3}{2})}{2n}\right)\\ &+\frac{m^2-\frac{1}{4}}{2\sin\theta}\,\frac{\cos\{(n+\frac{3}{2})\theta+\frac{1}{2}m\pi+\frac{1}{4}\pi\}}{n}+\frac{(\ldots)}{n^2}+\ldots\Bigg].\end{aligned}$$

This, though written in a slightly different form, agrees substantially with Hobson's result [(63), p. 490]. The necessity for the introduction of the quantities ϵ is well illustrated by an investigation in a recent paper by Bromwich*.

* Bromwich, *Proceedings of the London Mathematical Society*, Series 2, Vol. IV., pp. 212–213.

§ 47. *The asymptotic expansion of* $P_n^m(-\cos\theta)$, *when* $\epsilon_2 < \theta < \pi - \epsilon_2$.

By § 34, we have

$$\frac{-\iota\pi\sin(m+n)\pi}{\sin n\pi} P_n^m(-\cos\theta)$$
$$= G_n^m(e^{\iota\theta})\, e^{-(n+1)\pi\iota - \frac{1}{2}m\pi\iota} - G_n^m(e^{-\iota\theta})\, e^{(n+1)\pi\iota + \frac{1}{2}m\pi\iota}.$$

Hence

$$-\iota\sqrt{(\pi n)}\, n^{-m} P_n^m(-\cos\theta) = e^{(n+1)\iota(\theta-\pi) - \frac{1}{2}m\pi\iota}(1-e^{2\iota\theta})^{-\frac{1}{2}} \times \left[1 + \frac{{}^m C_1(\theta)}{n} + \ldots\right]$$
$$- e^{-(n+1)\iota(\theta-\pi) + \frac{1}{2}m\pi\iota}(1-e^{-2\iota\theta})^{-\frac{1}{2}}\left[1 + \frac{{}^m C_1(-\theta)}{n} + \ldots\right]$$
$$= e^{(n+1)\iota(\theta-\pi) - \frac{1}{2}m\pi\iota - \frac{1}{2}\iota\theta + \frac{1}{4}\pi\iota}(2\sin\theta)^{-\frac{1}{2}}\left\{1 + \frac{(\cdot\cdot)}{n} + \ldots\right\}$$
$$- e^{-(n+1)\iota(\theta-\pi) + \frac{1}{2}m\pi\iota + \frac{1}{2}\iota\theta - \frac{1}{4}\pi\iota}(2\sin\theta)^{-\frac{1}{2}}\left\{1 + \frac{(\cdot\cdot)}{n} + \ldots\right\}.$$

We therefore have

$$P_n^m(-\cos\theta) = -\sqrt{\left(\frac{2}{n\pi\sin\theta}\right)} \times n^m\left[\sin\left\{(n+\tfrac{1}{2})(\theta-\pi) - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi\right\} + \frac{(\cdot\cdot)}{n} + \ldots\right].$$

The approximation can, of course, be carried to any order should necessity arise.

§ 48. *The asymptotic value of* $P_n(x)$, *when* n *is a positive integer,* $|n|$ *is large, and* $|\xi| < 1 - \epsilon_1$.

In our previous investigation of the asymptotic expansion of $P_n^m(x)$, when $|n|$ is large, we have been continuously limited by the condition $|\arg(\pm n)| < \pi - \epsilon$, where ϵ is finite, small, and independent of n. It happens, however, that, in the fundamental expansion of $1/(y-x)$ in terms of Legendre functions, as well as in other investigations, we need the asymptotic value of $P_n(x)$, when n is a positive integer and x can take general complex values.

If $|\xi|<1-\epsilon_1$, *where* ϵ_1 *is a finite positive quantity, however small, independent of* n, *we will show that, when* n *is a positive integer,*

$$P_n(x)=\frac{\xi^{-n}}{\sqrt{(n\pi)}}(1-\xi^2)^{-\frac{1}{4}}+\frac{J(n)}{n^{\frac{3}{4}}},$$

where $|\arg\xi|<\pi$, $|\arg(1-\xi^2)|<\pi$, *and* $|J(n)|$ *can be made as small as we please by taking* $|n|$ *greater than a sufficiently large positive quantity.*

Heine (*loc. cit.*, p. 174) gives this result, but his proof lacks adequate rigour; the first satisfactory investigation appears to have been given by Darboux*.

When n is a positive integer one of the two series in § 29 vanishes, and we have, when $|\xi|<1$,

$$P_n(x)=\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)\,\Gamma(\frac{1}{2})}\xi^{-n}F\{\tfrac{1}{2},-n;\ \tfrac{1}{2}-n;\ \xi^2\}.$$

Hence

$$P_n(x)=-\frac{1}{2\pi\iota}\xi^{-n}\int\frac{\Gamma(\frac{1}{2}+n-s)\,\Gamma(-s)}{\Gamma(1+n-s)\,\Gamma(\frac{1}{2}-s)}\xi^{2s}ds\ldots(A),$$

the integral being taken round a finite contour which embraces the points $s=0, 1, 2, \ldots, n$. The subject of integration is, of course, a quotient of two polynomials in s.

Suppose now that $\xi=re^{\iota\theta}$, where $|\theta|<\pi$, $|r|<1-\epsilon_1$, and that $s=\sigma_1+\iota\sigma_2$, where σ_1 and σ_2 are real.

Then $$|\xi^s|=\exp\{-\sigma_1\log(1/r)-\theta\sigma_2\}.$$

Now, if $r<1-\epsilon_1$, $\log 1/r$ is positive; hence, under this limitation, we can always find an acute angle Φ', such that $\tan\Phi'=\dfrac{\log(1/r)}{|\theta|}$. If now we take two lines, making an angle Φ with the positive direction of the real axis, where $\Phi<\Phi'$, at a large distance ρ from the origin along these lines $|\xi^s|$ will behave like $\exp\{-\eta\rho\}$, where $\eta>0$.

Therefore it is evident that in the equality (A) we may take the contour of the integral to be an open contour as in fig. 9, coinciding with the two lines just drawn at a little distance from the origin and embracing this point.

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* Darboux, *Liouville*, Séries 3, T. 4, p. 22.

Now since n is a large positive integer, $|n-s|$ will always be large on this contour if Φ be greater than a finite positive quantity ϵ independent of n. And it is always possible to choose such a value of Φ if $|\xi|<1-\epsilon_1$, where ϵ_1 is a finite positive quantity, however small, independent of n.

Since $|n-s|$ is large we have

$$\frac{\Gamma(\frac{1}{2}+n-s)}{\Gamma(1+n-s)}=\frac{1+J_1(n-s)}{(n-s)^{\frac{1}{2}}},$$

where $|J_1(n-s)|$ can be made as small as we please by taking $|n|$ sufficiently large, and $|\arg(n-s)|<\pi$.

Hence

$$P_n(x)=-\frac{1}{2\pi\iota}\xi^n\int\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}-s)(n-s)^{\frac{1}{2}}}\xi^{2s}ds+\frac{J_2(n)}{n^{\frac{1}{2}}},$$

where $|J_2(n)|$ tends uniformly to zero as $|n|$ tends to infinity.

Therefore, since $|\xi|<1-\epsilon_1$, we have, as in § 39,

$$P_n(x)=-\frac{1}{2\pi\iota}\xi^{-n}\int\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}-s)\sqrt{n}}\xi^{2s}ds+\frac{J(n)}{n^{\frac{1}{2}}},$$

where $|J(n)|$ possesses a similar property.

Finally therefore

$$P_n(x)=\frac{\xi^{-n}}{\sqrt{(\pi n)}}(1-\xi^2)^{-\frac{1}{4}}+\frac{J(n)}{n^{\frac{1}{2}}},$$

and we have the given result under the assigned conditions.

The complete expansion given by Darboux (*loc. cit.*, p. 39) may be at once obtained.

An extension of the previous theorem may easily be found.

Let $|\xi|$ *be less than* $1-\epsilon_1$, *where* ϵ_1 *is a positive quantity, however small, independent of* n. *Suppose that* $|m|$ *is finite, that* $|n|$ *is large,* $|J(n)|$ *finite, and that* n *is not equal to half an odd integer. If, further,* $|\arg\xi|<\pi$, $|\arg(1-\xi^2)|<\pi$, *we have the asymptotic equality*

$$P_n^m(x)=\frac{\Gamma(1+n+m)}{\Gamma(\frac{1}{2})\Gamma(1+n-m)}\left\{\frac{\xi^{-n}(1-\xi^2)^{-\frac{1}{4}}}{n^{m+\frac{1}{2}}}+\frac{J(n)}{n^{m+\frac{1}{2}}}\right\},$$

where $|J(n)|$ *tends uniformly to zero with* $1/|n|$. This theorem is complementary to the results of § 45.

By § 29, if n be not equal to half an odd integer,

$$P_n^m(x) = \frac{\Gamma(-\frac{1}{2}-n)}{\Gamma(\frac{1}{2})\Gamma(-m-n)}(1-\xi^2)^{-m}\xi^{n+1}F\{\tfrac{1}{2}-m,\ 1+n-m;\ \tfrac{3}{2}+n;\ \xi^2\}$$

$$+\frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(n+1-m)}(1-\xi^2)^{-m}\xi^{-n}F\{\tfrac{1}{2}-m;\ -n-m;\ \tfrac{1}{2}-n;\ \xi^2\}.$$

Therefore, by the result of § 25,

$$P_n^m(x) - \frac{\tan n\pi}{\pi}\,Q_n^m(x)$$

$$= \frac{\Gamma(n+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(n+1-m)}(1-\xi^2)^{-m}\xi^{-n}\,F\{\tfrac{1}{2}-m,\ -n-m;\ \tfrac{1}{2}-n;\ \xi^2\}$$

$$= \frac{\Gamma(1+n+m)}{\Gamma(1+n-m)}\,\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}(1-\xi^2)^{-m}\xi^{-n}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(-s)\,\Gamma(\frac{1}{2}+n-s)}{\Gamma(\frac{1}{2}+m-s)\,\Gamma(1+n+m-s)}\,\xi^{2s}\,ds\ldots(1).$$

In the last integral the contour closely embraces the positive half of the real axis, and includes the sequence of poles 0, 1, 2, ... of the subject of integration. Moreover it is indented so as to exclude the sequence of poles $n+\frac{1}{2}$, $n+\frac{3}{2}$, ..., and such indentation is always possible if n be not equal to half an odd integer. If, however, we include this last sequence, we evidently only add terms which, when $|n|$ is large, are of lower exponential order than those which already exist. We can then, without invalidating our final result, suppose that such inclusion is made.

Again, by § 40, under conditions wider than those of the present enunciation, we have the asymptotic formula

$$Q_n^m(x)\frac{\tan n\pi}{\pi} = \frac{\sin\pi(n+m)}{\sqrt{(\pi n)}\cos\pi n}\,\xi^{n+1}(1-\xi^2)^{-\frac{1}{2}}n^m\left\{1+\frac{{}^mC_1}{n}+\ldots\right\}.$$

As will appear in the sequel, these terms are of lower exponential order than others which arise from the equality (1). We may therefore neglect them.

As in the simple case just considered, for which $m=0$, we may now open out the contour of the integral, if $|\xi|<1-\epsilon_1$, so as to make a small, but finite, angle with the positive half of the real axis on either side of that axis. On the new

contour we may employ the asymptotic formula for the gamma-function, when $|n-s|$ is large. We find therefore

$$P_n^m(x)\frac{\Gamma(1+n-m)\,\Gamma(\frac{1}{2})}{\Gamma(1+n+m)\,\Gamma(m+\frac{1}{2})}(1-\xi^2)^m\xi^n$$
$$=-\frac{1}{2\pi\iota}\int\frac{\Gamma(-s)\,\xi^{2s}\,ds}{\Gamma(\frac{1}{2}+m-s)\,(n-s)^{\frac{1}{2}+m}}+\frac{J_1}{n^{m+\frac{1}{2}}},$$

when $|J_1|$ tends uniformly to zero with $1/|n|$.

We may now close up the contour so as to embrace the positive half of the real axis and the singularity $s=n$, and by Cauchy's theorem we get

$$\sum_{t=0}^{\nu}\frac{(-)^t\,\xi^{2t}}{t!\,\Gamma(\frac{1}{2}+m-t)\,(n-t)^{\frac{1}{2}+m}}+\frac{J_2}{n^{\frac{1}{2}+m}},$$

where ν is an integer such that $\nu<R(n)\leq\nu+1$, and we have neglected terms of lower exponential order, when $|n|$ is large, than those retained.

And, neglecting similar terms once more, we find

$$\frac{1}{\Gamma(\frac{1}{2}-m)\,\Gamma(\frac{1}{2}+m)}\sum_{t=0}^{\infty}\frac{\Gamma(\frac{1}{2}-m+t)\,\xi^{2t}}{t!\,n^{m+\frac{1}{2}}}+\frac{J_3}{n^{\frac{1}{2}+m}}.$$

We finally have the given result.

Evidently any number of terms of the asymptotic series may be obtained in the same way by further algebra.

Asymptotic expansions for $P_n^m(x)$ and $Q_n^m(x)$, *when* $|m|$ *is large.*

§ 49. We now proceed to consider the nature of the asymptotic formulæ for $P_n^m(x)$ and $Q_n^m(x)$, when $|m|$ is large, n and x having general complex values. The problem will be solved by methods very analogous to those which have just been used in the consideration of asymptotic formulæ when $|n|$ is large. The results are of importance in that by their means we can determine the area of convergence of the expansion of $\{x+\sqrt{(x^2-1)}\cos\phi\}^n$, in cosines of multiples of ϕ, when n, x, and ϕ have general complex values; and, as will be shown later, this expansion can be made fundamental in proving the addition formula for Laplace's functions

$$P_n[xx_1+\sqrt{\{(x^2-1)\,(x_1^2-1)\}}\cos\phi],$$
$$Q_n[xx_1+\sqrt{\{(x^2-1)\,(x_1^2-1)\}}\cos\phi],$$

for general complex values of n, x, and ϕ.

The asymptotic expansion of $P_n^m(x)$, *when* $|\arg(-m)| < \pi - \epsilon$, $R(x) > \epsilon_1$, *and* x *not real and less than or equal to unity.*

By the asymptotic expansion of the gamma function, we know that when $|x|$ is large and $|\arg x| < \pi$,

$$\frac{\Gamma(x-n)}{\Gamma(x+1)} = \frac{1}{x^{n+1}} \exp\left\{ \sum_{r=1}^{N} \frac{(-)^{r-1}}{r x^r} [S_r(-n) - S_r(1)] + \frac{{}_1J_N(x)}{x^N} \right\},$$

where $S_r(n)$ is the r^{th} Bernoullian function of n, and $|{}_1J_N(x)|$ tends uniformly to zero as $|x|$ tends to infinity.

If then we determine the functions $L_r (r \leq N)$ by the expansion

$$\exp\left\{ \sum_{r=1}^{N} \frac{(-)^{r-1}}{r x^r} [S_r(-n) - S_r(1)] \right\} = \sum_{r=0} \frac{L_r}{x^r},$$

we shall have, when $|x|$ is large and $|\arg x| < \pi$,

$$\frac{\Gamma(x-n)}{\Gamma(x+1)} = \frac{1}{x^{n+1}} \left\{ \sum_{r=0}^{N} \frac{L_r}{x^r} + \frac{{}_2J_N(x)}{x^N} \right\},$$

where $|{}_2J_N(x)|$ tends uniformly to zero as $|x|$ tends to infinity.

We may now show that if $|\arg(-m)| < \pi - \epsilon$, *where* ϵ *is, however small, finite, positive, and independent of* m, *if, further,* $R(x) > \epsilon_1$ *and* x *is not real and less than or equal to unity, we have, when* $|m|$ *is large, the asymptotic expansion*

$$P_n^m(x) = \frac{1}{\Gamma(-n)\,\Gamma(-m-n)} \left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}m} \left(\frac{2}{1+x}\right)^{-n} \times \left[\sum_{q=0}^{N} \frac{\Gamma(q+n+1)}{(-m)^{n+1+q}} \chi_q(x) + \frac{J(m)}{m^{n+1+N}} \right],$$

where

$$\chi_q(x) = \sum_{t=0}^{\infty} \frac{\Gamma(t-n)}{\Gamma(t+1)} \left(\frac{x-1}{x+1}\right)^t \sum_{p=0}^{q} \frac{(-)^p t^p L_{q-p}}{p!\,\Gamma(q-p+n+1)},$$

and $|J(m)|$ *tends uniformly to zero as* $|m|$ *tends to infinity.*

By § 7 we have, if x does not lie on the cross cut $-\infty$ to 1,

$$P_n^{-m}(x)\,\Gamma(-n)\,\Gamma(m-n) \left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}m} \left(\frac{2}{1+x}\right)^n = -\frac{1}{2\pi i} \int \frac{\Gamma(-s)\,\Gamma(s-n)\,\Gamma(s+m-n)}{\Gamma(1+s+m)} \left(\frac{1-x}{1+x}\right)^s ds \ldots (1).$$

If $|\arg m| < \pi - \epsilon$, and $\left|\frac{1-x}{1+x}\right| < 1$, so that $R(x) > 0$, we may take the contour to embrace closely the positive half of the real axis, and then it will exclude the sequences n, $n-1$, $n-2$, ..., and $n-m$, $n-m-1$, ... Moreover $|s+m|$ will be large at all points on the contour. We may therefore employ the asymptotic expansion for $\frac{\Gamma(s+m-n)}{\Gamma(1+s+m)}$, and the integral is equal to

$$-\frac{1}{2\pi\iota}\sum_{r=0}^{N} L_r \int \frac{\Gamma(-s)\,\Gamma(s-n)}{(s+m)^{n+r+1}} \left(\frac{1-x}{1+x}\right)^s ds$$

$$-\frac{1}{2\pi\iota}\int \frac{\Gamma(-s)\,\Gamma(s-n)\,{}_2J_N(m+s)}{(m+s)^{N+n+1}} \left(\frac{1-x}{1+x}\right)^s ds$$

$$= \sum_{r=0}^{N} \frac{L_r}{m^{n+r+1}} \sum_{t=0}^{\infty} \frac{\Gamma(t-n)}{\Gamma(t+1)\,(1+t/m)^{n+r+1}} \left(\frac{x-1}{x+1}\right)^t + \frac{{}_3J_N}{m^{N+n+1}},$$

where $|{}_3J_N|$ tends uniformly to zero as $|m|$ tends to infinity provided $R(x) > \epsilon_1$, where ϵ_1 has the same properties as ϵ.

The last expansion is equal to

$$\sum_{r=0}^{N} \frac{L_r}{m^{n+r+1}} \sum_{t=0}^{\infty} \frac{\Gamma(t-n)}{\Gamma(t+1)} \left(\frac{x-1}{x+1}\right)^t$$

$$\times \left(-\frac{1}{2\pi\iota}\right) \int \frac{\Gamma(\phi+n+r+1)\,\Gamma(-\phi)}{\Gamma(n+r+1)} \left(\frac{t}{m}\right)^{\phi} d\phi + \frac{{}_5J_N}{m^{N+n+1}},$$

the contour being, apart from a loop if necessary, parallel to the imaginary axis. Hence, as in the previous investigation of § 39, we have

$$\sum_{r=0}^{N} \frac{L_r}{m^{n+r+1}} \sum_{t=0}^{\infty} \frac{\Gamma(t-n)}{\Gamma(t+1)} \left(\frac{x-1}{x+1}\right)^t$$

$$\times \sum_{p=0}^{N} \frac{\Gamma(p+n+r+1)}{\Gamma(n+r+1)} \frac{(-t)^p}{p!\,m^p} + \frac{{}_4J_N}{m^{N+n+1}}$$

$$= \sum_{q=0}^{N} \frac{\Gamma(q+n+1)}{m^{n+1+q}} \chi_q(x) + \frac{J_N(m)}{m^{N+n+1}}.$$

Change now m into $-m$, and we have the given result under the assigned conditions.

§ 50. *Suppose that* $(x-1)/(x+1) = re^{i\theta}$ *where* $\varepsilon_0 < |\theta| < \pi$. *Then we may show that the previous expansion for* $P_n^m(x)$ *is also valid:—*

(1) *for all values of* $\log r$ *which do not tend to infinity with* $|m|$, *provided* $|\arg(-m)| \leqslant \frac{1}{2}\pi + \eta$, *where* η *is a sufficiently small positive quantity, independent of* $|m|$; *or*

(2) *if* $\log r + \epsilon_1 < |\theta| \tan |\arg m|$, *and* $\frac{1}{2}\pi \geqslant |\arg m| > \epsilon$.

When $r < 1$, $\log r$ *is negative: the inequalities are satisfied, and we have substantially the case of* § 49. *When* $r > 1$, *the series for* $\chi_q(x)$ *is divergent, and we must replace it by the contour integral*

$$\chi_q(x) = -\frac{1}{2\pi\iota}\int \Gamma(s-n)\Gamma(-s)\left(\frac{1-x}{1+x}\right)^s \sum_{p=0}^{q} \frac{(-)^p s^p L_{q-p}}{p!\,\Gamma(q-p+n+1)}\,ds.$$

When $r \geqslant 1$, *we take the contour of the integral, as in* § 48, *so that at a large distance from the origin it coincides with two straight lines drawn from the origin and making an angle, in case* (1), *equal to* $\frac{1}{2}\pi - \eta - \varepsilon_2$, *and, in case* (2), *equal to* $|\arg m| - \varepsilon_3$, *where* $\varepsilon_3 < \varepsilon$, *with the positive direction of the real axis. All the quantities* ε *are finite positive quantities, as small as we please, independent of* $|m|$.

We note in the first place that the inequalities imposed upon θ uniquely prescribe $\left(\frac{x-1}{x+1}\right)^{\pm\frac{1}{2}m}$, and, further, that they exclude all points on the real axis.

In the second place we notice that, if we replace $-m$ by m, we may write the contour integral (1) of § 49 in the form

$$\frac{1}{2\pi\iota}\int \frac{\Gamma(s-n)}{\Gamma(s+1)}\,\frac{\Gamma(s-m-n)}{\Gamma(s-m+1)}\,\frac{\pi}{\sin\pi s}\left(\frac{1-x}{1+x}\right)^s ds.$$

In this formula $\left|\arg\frac{1-x}{1+x}\right| < \pi$,

and therefore $\arg\frac{1-x}{1+x} = \theta \pm \pi$,

the upper or lower sign being taken as θ is negative or positive.

Therefore, if $s = ke^{i\phi}$, and k be large,

$$\left| \frac{\pi}{\sin \pi s} \left(\frac{1-x}{1+x} \right)^s \right|$$

$$\lesssim 2\pi \exp k \{ -\pi |\sin\phi| + \cos\phi \log r + |\sin\phi| (\pi - |\theta|) \}$$

$$\lesssim 2\pi \exp [-k \{ |\sin\phi| |\theta| - \cos\phi \log r \}].$$

Hence, if the ϵ's are suitably chosen, the integrals are always, under the assigned conditions, convergent. Moreover, on the assigned contours, $|s-m|$ is always large and $|\arg(-s/m)| < \pi$.

As before, the integral may be written

$$-\frac{1}{2\pi\iota} \sum_{r=0}^{N} \frac{L_r}{(-m)^{n+r+1}} \int \frac{\Gamma(-s)\,\Gamma(s-n)}{(1-s/m)^{n+r+1}} \left(\frac{1-x}{1+x} \right)^s ds + \frac{{}_3J_N}{(-m)^{N+n+1}}$$

$$= \left(-\frac{1}{2\pi\iota} \right)^2 \sum_{r=0}^{N} \frac{L_r}{(-m)^{n+r+1}} \int \left(\frac{1-x}{1+x} \right)^s \Gamma(-s)\,\Gamma(s-n)$$

$$\times \int \frac{\Gamma(\phi+n+r+1)}{\Gamma(n+r+1)} \Gamma(-\phi) \left(\frac{-s}{m} \right)^{\phi} d\phi\, ds + \frac{{}_3J_N}{(-m)^{N+n+1}}$$

$$= -\frac{1}{2\pi\iota} \sum_{r=0}^{N} \frac{L_r}{(-m)^{n+r+1}} \int \left(\frac{1-x}{1+x} \right)^s (\Gamma - s)\,\Gamma(s-n)\, ds$$

$$\times \sum_{p=0}^{N} \frac{\Gamma(p+n+r+1)}{\Gamma(n+r+1)} \frac{s^p}{p!\, m^p}$$

$$-\left(\frac{1}{2\pi\iota} \right)^2 \sum_{r=0}^{N} \frac{L_r}{(-m)^{n+r+1}} \int \left(\frac{1-x}{1+x} \right)^s \Gamma(-s)\,\Gamma(s-n)\, ds$$

$$\times \int_L \frac{\Gamma(\phi+N+n+r+1)}{\Gamma(n+r+1)} \Gamma(-\phi-N) \left(\frac{-s}{m} \right)^{\phi+N} d\phi + \frac{{}_3J_N}{(-m)^{N+n+1}},$$

the contour L being apart from a loop, if necessary, parallel to the imaginary axis, and cutting the real axis between the points 0 and 1.

As in § 29 the previous expression is equal to

$$-\frac{1}{2\pi\iota} \int \left(\frac{1-x}{1+x} \right)^s \Gamma(s-n)\,\Gamma(-s)\, ds$$

$$\times \sum_{r=0}^{N} \sum_{p=0}^{N} \frac{L_r\,\Gamma(p+n+r+1)}{(-m)^{n+p+r+1}\,\Gamma(n+r+1)} \frac{(-)^p s^p}{p!} + \frac{{}_2J_N}{(-m)^{N+n+1}},$$

where, for any assigned finite value of N, $|{}_2J_N|$ tends uniformly to zero with $1/|m|$.

And this yields the given result.

§ 51. *We proceed now to obtain the dominant terms of the previous asymptotic equality. We will show that, if x be not on the cross cut, $-\infty$ to $+1$, and, further, if either*

$$(1) \qquad |\arg m| < \pi - \varepsilon \text{ and } R(x) > \varepsilon_1,$$

or (2) *if $\epsilon_0 < |\theta| < \pi$, $|\arg m| \leq \frac{1}{2}\pi + \eta$ where η is a sufficiently small positive quantity, independent of $|m|$, and, for all finite values of r, if $\log r$ do not tend to infinity with $|m|$,*

or (3) *if $\epsilon_0 < |\theta| < \pi$, $\frac{1}{2}\pi \geq |\arg(-m)| > \epsilon$,*

and

$$\log r + \epsilon_1 < |\theta| \tan |\arg(-m)|,$$

r and θ being defined by the equality $(x-1)/(x+1) = re^{i\theta}$, and the ϵ's being suitably chosen positive quantities, as small as we please, independent of $|m|$; we have the asymptotic equality

$$P_n^{-m}(x) = \frac{1}{\Gamma(m-n)} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \frac{1}{m^{n+1}} \left\{1 + \frac{n(n+1)x}{2m} + \frac{(\cdot\cdot)}{m^2} + \cdot\cdot\right\}.$$

Since $S_1(a) = \frac{1}{2}(a^2 - a)$, we have $L_0 = 1$, $L_1 = \frac{1}{2}\{n(n+1)\}$.

Hence, from the previous result under the assigned conditions, if $R(x) > \epsilon_1$,

$$P_n^{-m}(x)\,\Gamma(-n)\,\Gamma(m-n)\left(\frac{x-1}{x+1}\right)^{-\frac{1}{2}m}\left(\frac{2}{1+x}\right)^n$$

$$= \sum_{t=0}^{\infty} \frac{\Gamma(t-n)}{\Gamma(t+1)} \left(\frac{x-1}{x+1}\right)^t \left\{\frac{1}{m^{n+1}} + \frac{n(n+1)}{2m^{n+2}} + \ldots\right\}$$

$$+ \sum_{t=0}^{\infty} \frac{\Gamma(t-n)}{\Gamma(t+1)} \left(\frac{x-1}{x+1}\right)^t \frac{\Gamma(n+2)}{m^{n+2}} \frac{(-t)}{\Gamma(n+1)} + \ldots$$

$$= \frac{\Gamma(-n)}{m^{n+1}} \left(1 - \frac{x-1}{x+1}\right)^n \left\{1 + \frac{n(n+1)}{2m}\right\}$$

$$- \frac{n+1}{m^{n+2}} \left(\frac{x-1}{x+1}\right) \sum_{t=0}^{\infty} \frac{\Gamma(t+1-n)}{\Gamma(t+1)} \left(\frac{x-1}{x+1}\right)^t + \ldots$$

$$= \frac{\Gamma(-n)}{m^{n+1}} \left(\frac{2}{1+x}\right)^n \left\{1 + \frac{n(n+1)x}{2m} + \frac{(\cdot\cdot)}{m^2} + \ldots\right\}.$$

When $R(x) < \varepsilon_1$, we must work with the equivalent contour integrals; we evidently arrive at the same result.

We thus have the given theorem.

The result may be written in a slightly different form.

For

$$\Gamma(m-n) = m^{m-n-\frac{1}{2}}\sqrt{(2\pi)}\, e^{-m}\left\{1 + \frac{n(n+1)}{2m} + \ldots\right\}.$$

Hence, under the assigned conditions,

$$P_n^{-m}(x) = \frac{m^{-m-\frac{1}{2}}}{\sqrt{(2\pi)}}\, e^m \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\left\{1 + \frac{n(n+1)}{m}\left(\frac{x-1}{2}\right) + \frac{(\cdot\cdot)}{m^2} + \ldots\right\}.$$

§ 52. *The case when m is a positive integer.*

We may now show that when m is a large positive integer, when n has general complex values, and for all values of x of finite modulus, except for those on the cross cut, $-\infty$ to 1, or at a distance from the cross cut, $-\infty$ to -1, which vanishes with $1/m$, we have the asymptotic equality

$$P_n^m(x) = (-)^{m-1}\sqrt{\left(\frac{2}{\pi m}\right)}\sin n\pi\, \frac{m^m}{e^m}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \times \left\{1 + \frac{n(n+1)}{m}\{\tfrac{1}{2}(x+1)\} + \frac{(\cdot\cdot)}{m^2} + \ldots\right\},$$

wherein $\left|\arg\dfrac{x-1}{x+1}\right| < \pi$.

For, when m is a positive integer and x does not lie on the cross cut, we have, by § 13,

$$P_n^{-m}(x) = P_n^m(x)\,\frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}.$$

Hence, under the assigned conditions,

$$P_n^m(x) = \Gamma(n+m+1)\,\frac{\sin(m-n)\pi}{\pi}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\frac{1}{m^{n+1}} \times \left\{1 + \frac{n(n+1)x}{2m} + \frac{(\cdot\cdot)}{m^2} + \ldots\right\},$$

which yields the given result.

§ 53. The result of § 51 gives the asymptotic expansion of $P_n^m(x)$, for *all* values of x except those on or very

near the cross cut, only when $|\arg(-m)| \lesseqgtr \frac{1}{2}\pi + \eta$, where η is a sufficiently small positive quantity, independent of m. It is evidently important to try and obtain the analogous expansion, when $|\arg(m)| \lesseqgtr \frac{1}{2}\pi + \eta$.

We proceed to show that, if $\epsilon_0 < |\theta| < \pi$, *and if either*

(1) $|\arg m| \lesseqgtr \frac{1}{2}\pi + \eta$, *and* r *has any value such that* $\log r$ *does not tend to infinity with* m,

or (2) *if* $\qquad \frac{1}{2}\pi \gtrless |\arg(-m)| > \epsilon$,

and $\qquad |\log r| + \epsilon_1 < |\theta| \tan |\arg(-m)|$,

the ϵ*'s being finite positive quantities, as small as we please, independent of* m, *and* x *not lying on the real axis, we have, when* $|m|$ *is large, the asymptotic equality*

$$\frac{\pi P_n^m(x)}{\Gamma(1+m+n)} = \sin \pi m \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} m^{-n-1} \left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}$$

$$- \sin \pi n \, e^{\mp \pi \iota m} \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} m^{-n-1} \left\{1 + \frac{n(n+1)x}{2m} + \ldots\right\} \ldots (A),$$

the upper or lower sign being taken as $I(x)$ *is positive or negative.*

We have, by the result obtained in § 12,

$$\frac{\pi P_n^m(x)}{\Gamma(1+m+n)}$$

$$= P_n^{-m}(-x)\,\Gamma(m-n)\sin m\pi - e^{\mp \pi \iota m} \sin n\pi\, \Gamma(m-n)\, P_n^{-m}(x).$$

Combine this with the result of § 51 and we have the given theorem.

It is evident that the previous formula yields various expansions according to the position of x and the sign of the imaginary part of m. Should necessity arise the various cases could be discussed after the manner of § 45.

One question is, however, of particular interest. In § 51 we have shown that, if either

(3) $\qquad R(x) > \epsilon_1$, and $|\arg(-m)| < \pi - \epsilon$,

or (4) $\epsilon_0 < |\theta| < \pi$, $|\arg(-m)| \lesseqgtr \frac{1}{2}\pi + \eta$, and for all values of r, such that $\log r$ is finite and does not tend to infinity with $|m|$,

or (5) $\qquad \varepsilon_0 < |\theta| < \pi, \quad \tfrac{1}{2}\pi \gtrless |\arg m| > \varepsilon,$

and $\qquad \log r + \varepsilon_1 < |\theta| \tan |\arg m|,$

we have

$$P_n^m(x)\Gamma(-m-n) = \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} (-m)^{-n-1}\left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\} \ldots (B).$$

This result must therefore be the same as that furnished by the equality (A), under conditions common to (1), or (2), and (3), (4), or (5).

We proceed to verify that this is the case.

We will limit the discussion to the case when $\arg m$ is positive.

Then evidently the equality (B) may be written

$$\frac{\pi P_n^m(x)}{\Gamma(1+m+n)} = \tfrac{1}{2}\iota\, e^{-\pi\iota m}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} m^{-n-1}\left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}.$$

The equality (A) becomes

$$\frac{\pi P_n^m(x)}{\Gamma(1+m+n)} = \tfrac{1}{2}\iota\, e^{-\pi\iota m}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} m^{-n-1}\left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}$$

$$- \sin \pi n\, e^{\mp \pi\iota m}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} m^{-n-1}\left\{1 + \frac{n(n+1)x}{2m} + \ldots\right\},$$

the upper or lower sign being taken as $I(x)$ is positive or negative.

We have then to show that, in all cases in which the conditions (1) and (2) are included among the conditions (3), (4), or (5), the first term of the last asymptotic equality is dominant compared with the second.

Put $m = \mu + \iota\nu$, where μ and ν are real and ν is positive.

Then the relative magnitudes of the moduli of the first and second terms in the last asymptotic equality are in the ratio

$$e^{\nu(\pi+\frac{1}{2}\theta)-\frac{1}{2}\mu\log r} : e^{\pm\pi\nu-\frac{1}{2}\theta\nu+\frac{1}{2}\mu\log r},$$

the upper or lower sign being taken as θ is positive or negative.

We must therefore have

$$\mu \log r < |\theta|\,|\nu| \text{ and } \mu \log r < \{2\pi - |\theta|\}\,|\nu|,$$

whenever $\varepsilon < |\arg m| < \tfrac{1}{2}\pi$, and $\log r + \varepsilon_1 < |\theta| \tan |\arg m|$. And this is certainly true. A similar investigation holds when $\arg m$ is negative. The verification is complete.

Summing up our results, we see that it is convenient to use the formula (A), when $|\arg m| \leq \frac{1}{2}\pi + \eta$: in this case x must not be on the real axis or at a distance from the parts of it comprised by $-\infty$ to -1, or 1 to $+\infty$, which vanishes with $1/m$. Such restrictions are sufficient; we have not proved that they are all necessary. It is convenient to use the formula (B), when $\pi \geq |\arg m| \geq \frac{1}{2}\pi - \eta$; in this case x must not lie on the cut $-\infty$ to 1, or at a distance from that part of it comprised by $-\infty$ to -1 which vanishes with $1/m$. We notice also that, in either case, as regards points on the cross cut for which $-1+\varepsilon_1 < x < 1-\varepsilon_1$, where $\varepsilon_1 > 0$ and independent of m, we are at liberty to take $x = \cos\theta \pm \varepsilon\iota$ and make ε tend to zero without invalidating the asymptotic nature of the formula.

§ 54. *The asymptotic expansion of* $P_n^{-m}(x)$ *when* $|m|$ *is large and* x *lies on the cross cut so that* $-1+\varepsilon_1 < x < 1-\varepsilon_1$.

Suppose now that $x = \cos\theta$, where $\varepsilon_2 < \theta < \pi - \varepsilon_2$.

We will show that, when $|m|$ is large, we have the asymptotic formula

$$P_n^{-m}(\cos\theta) = \frac{1}{\Gamma(m-n)} \frac{(\tan\frac{1}{2}\theta)^m}{m^{n+1}} \times \left\{1 + \frac{n(n+1)}{2m}\cos\theta + \frac{(\cdot\cdot)}{m^2} + \ldots\right\},$$

when $\quad |\arg m| < \pi - \varepsilon, \quad$ if $\varepsilon_2 < \theta < \frac{1}{2}\pi - \varepsilon_2$,

and when $\quad |\arg m| \leq \frac{1}{2}\pi + \eta, \quad$ if $\varepsilon_2 < \theta < \pi - \varepsilon_2$.

For, by § 13, in the limit when $\epsilon = 0$,

$$P_n^{-m}(\cos\theta + \varepsilon\iota) = P_n^{-m}(\cos\theta)\, e^{m\frac{1}{2}\pi\iota}.$$

Also $\quad (\cos\theta + \varepsilon\iota - 1)^{\frac{1}{2}m} = (1-\cos\theta)^{\frac{1}{2}m} e^{\pi\iota\frac{1}{2}m}$.

The given result therefore follows by the result of § 51.

Similarly, by § 53, we may show that, if $|\arg m| \leq \frac{1}{2}\pi + \eta$, if $\varepsilon_2 < \theta < \pi - \varepsilon_2$, and $|n|$ is large, we have the asymptotic formula

$$\frac{\pi P_n^m(\cos\theta)}{\Gamma(1+m+n)} = \frac{\sin\pi m}{m^{n+1}} \cot^m\tfrac{1}{2}\theta \left\{1 - \frac{n(n+1)\cos\theta}{2m} + \ldots\right\} - \frac{\sin\pi n}{m^{n+1}} \tan^m\tfrac{1}{2}\theta \left\{1 + \frac{n(n+1)\cos\theta}{2m} + \ldots\right\}.$$

This result is equivalent to the preceding when

$$\varepsilon_2 < \theta < \tfrac{1}{2}\pi - \varepsilon_2,$$

and

$$\varepsilon < |\arg m| \leqslant \tfrac{1}{2}\pi + \eta.$$

Suppose next that m is a large positive integer and $-1+\varepsilon_1 < x < 1-\varepsilon_1$.

By § 13

$$P_n^{-m}(x) = (-)^m P_n^m(x) \frac{\Gamma(1+n-m)}{\Gamma(1+n+m)}.$$

Hence, by the previous result, if m be a large positive integer, $\varepsilon_2 < \theta < \pi - \epsilon_2$, we have, for general complex values of n, the asymptotic formula

$$P_n^m(\cos\theta) = -\frac{\sin\pi n}{\pi}\,\Gamma(1+n+m)\,\frac{(\tan\frac{1}{2}\theta)^m}{m^{n+1}} \times \left\{1 + \frac{n(n+1)}{2m}\cos\theta + \frac{(\cdot\cdot)}{m^2} + \ldots\right\}.$$

When n is a positive integer or zero this of course reduces to zero, since (p. 112)

$$P_n^m(\cos\theta) = 0 \text{ when } m > n,$$

m and n being positive integers.

§ 55. *The asymptotic expansion of $Q_n^m(x)$ when $|m|$ is large and x is not on the real axis.*

We will now show that, if $|m|$ is large and

$$\epsilon_0 < \left|\arg\frac{x-1}{x+1}\right| < \pi,$$

we have

$$2\,Q_n^m(x)\,\Gamma(-m-n)\,\frac{\sin n\pi}{\pi} = e^{\mp n\pi\iota}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m}\frac{1}{m^{n+1}} \times \left\{1 + \frac{n(n+1)\,x}{2m} + \ldots\right\} - \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m}\frac{1}{m^{n+1}}\left\{1 - \frac{n(n+1)\,x}{2m} + \ldots\right\},$$

when $|\arg m| \leqq \frac{1}{2}\pi + \eta$; and that the same expression is equal to

$$e^{\mp n\pi\iota}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \frac{1}{(-m)^{n+1}} \left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}$$

$$- \left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \frac{1}{(-m)^{n+1}} \left\{1 + \frac{n(n+1)x}{2m} + \ldots\right\},$$

when $|\arg(-m)| \leqq \frac{1}{2}\pi + \eta$, the upper or lower sign being taken as $I(x)$ is positive or negative.

By §§ 9 and 10, we have

$$2Q_n^{-m}(x)\,\Gamma(m-n)\frac{\sin n\pi}{\pi} = 2Q_n^{m}(x)\,\Gamma(-m-n)\frac{\sin n\pi}{\pi}$$

$$= \Gamma(m-n)\,P_n^{-m}(x)\,e^{\mp n\pi\iota} - \Gamma(m-n)\,P_n^{-m}(-x).$$

If, then, $|\arg m| \leqq \frac{1}{2}\pi + \eta$, and $\epsilon_0 < \left|\arg\frac{x-1}{x+1}\right| < \pi$, x therefore not lying on the real axis, we have, by the result of § 51,

$$2Q_n^{-m}(x)\,\Gamma(m-n)\frac{\sin n\pi}{\pi} = 2Q_n^{m}(x)\,\Gamma(-m-n)\frac{\sin n\pi}{\pi}$$

$$= e^{\mp n\pi\iota}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}m} \frac{1}{m^{n+1}} \left\{1 + \frac{n(n+1)x}{2m} + \ldots\right\}$$

$$- \left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \frac{1}{m^{n+1}} \left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}.$$

These two equalities evidently yield the given formula.

A detailed investigation of the various cases, in which one or other of the two series is dominant, can evidently be carried out as in § 45.

The work is tedious and the results, though easy to obtain, are not very elegant.

One particular case may be noticed. If $|\arg m| < \frac{1}{2}\pi - \epsilon$, x real and > 1, we have the asymptotic formulæ

$$2Q_n^{-m}(x)\,\Gamma(m-n)\frac{\sin n\pi}{\pi} = 2Q_n^{m}(x)\,\Gamma(-m-n)\frac{\sin n\pi}{\pi}$$

$$= -\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}m} \frac{1}{m^{n+1}} \left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}.$$

Corresponding formulæ evidently do not exist when x is real and < -1, for we are then on the cross cut for $Q_n^{\pm m}(x)$.

§ 56. *The asymptotic expansion of* $Q_n^m(x)$, *when* $|m|$ *is large and* x *is on the cross cut given by* $-1+\epsilon_1 < x < 1-\epsilon_1$.

We will now show that, when $|m|$ is large and $x=\cos\theta$, where $\epsilon_2 < \theta < \pi - \epsilon_2$, we have, if $|\arg m| \leq \frac{1}{2}\pi + \eta$,

$$2\,Q_n^m(x)\,\Gamma(-m-n)\,\frac{\sin n\pi}{\pi}$$

$$= \frac{\cos(\frac{1}{2}m-n)\,\pi}{m^{n+1}}\tan^m\tfrac{1}{2}\theta\left\{1+\frac{n(n+1)\cos\theta}{2m}+\ldots\right\}$$

$$-\frac{\cos\frac{1}{2}m\pi}{m^{n+1}}\cot^m\tfrac{1}{2}\theta\left\{1-\frac{n(n+1)\cos\theta}{2m}+\ldots\right\}.$$

The corresponding formula, when $|\arg(-m)| \leq \frac{1}{2}\pi+\eta$, can evidently be immediately obtained from this by the relation

$$Q_n^m(x)\,\Gamma(-m-n) = Q_n^{-m}(x)\,\Gamma(m-n).$$

The two formulæ give the same result when m is on the imaginary axis.

When $-1<x<1$, we have, by § 14, in the limit when $\epsilon=0$,

$$Q_n^m(x) = \tfrac{1}{2}\{Q_n^m(x+\epsilon\iota)+Q_n^m(x-\epsilon\iota)\}.$$

Therefore, by § 55, we have, if $-1+\epsilon_1<x<1-\epsilon_1$, $|m|$ large, $|\arg m| \leq \frac{1}{2}\pi+\eta$, the asymptotic formula

$$4\,Q_n^m(x)\,\Gamma(-m-n)\,\frac{\sin n\pi}{\pi}$$

$$= e^{(\frac{1}{2}m-n)\pi\iota}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}m}\frac{1}{m^{n+1}}\left\{1+\frac{n(n+1)x}{2m}+\ldots\right\}$$

$$-\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m}e^{-\frac{1}{2}m\pi\iota}\frac{1}{m^{n+1}}\left\{1-\frac{n(n+1)x}{2m}+\ldots\right\}$$

$$+e^{(n-\frac{1}{2}m)\pi\iota}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}m}\frac{1}{m^{n+1}}\left\{1+\frac{n(n+1)x}{2m}+\ldots\right\}$$

$$-\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}m}e^{m\pi\iota}\frac{1}{m^{n+1}}\left\{1-\frac{n(n+1)x}{2m}+\ldots\right\}.$$

This immediately yields the given result.

In particular, we notice that, if

$$\epsilon_2 < \theta < \tfrac{1}{2}\pi - \epsilon_2, \quad |\arg m| < \tfrac{1}{2}\pi - \epsilon,$$

and m is not an odd integer, we have, when $|m|$ is large, the asymptotic formula

$$2\,Q_n^m(x)\,\Gamma(-m-n)\,\frac{\sin n\pi}{\pi}$$

$$= -\cot^m \tfrac{1}{2}\theta\,\frac{\cos \tfrac{1}{2}m\pi}{m^{n+1}}\left\{1 - \frac{n(n+1)x}{2m} + \ldots\right\}.$$

The conclusion that, when m is an odd integer,

$$2\,Q_n^m(x)\,\Gamma(-m-n)$$

$$= (-)^{\frac{1}{2}(m-1)}\,\frac{\pi \tan^m \tfrac{1}{2}\theta}{m^{n+1}}\left\{1 + \frac{n(n+1)x}{2m} + \ldots\right\}.$$

could not be inferred without a separate investigation, for in this case the dominant series, which is all that our formulæ pretend to yield, vanishes, as it were, accidentally. This and further particular cases we leave to the reader.

Part III.

Various propositions in the theory of generalised Legendre functions.

§ 57. We now proceed to apply the fundamental contour integrals, introduced in Part I. of the present paper, to obtain some of the more important theorems relating to generalised Legendre functions. It is evident that the whole known theory can be built up in a very general and comparatively simple manner by the aid of such integrals. To develop such an investigation completely would, however, demand more space than can be legitimately conceded to a single paper. I therefore propose to be frankly eclectic, and to consider, on the one hand, such parts of the theory as are of peculiar importance, and, on the other hand, results for which the new methods lead to hitherto unsuspected generalisations of known formulæ. An example of the former category is the difference equation for the generalised Legendre functions: the proof

by the present methods is, as might be expected, peculiarly simple. Another example is the theorem that

$$\int_{-1}^{1} P_n^m(x)\, P_{n_1}^m(x) = 0,$$

when m, n, and n_1 are positive integers ($n_1 \neq n$), and

$$= \frac{2\,\Gamma(1+m+n)}{\Gamma(1+n-m)\,(2n+1)},$$

when $n = n_1$: the new methods enable us to dispense with the usual reduction formula derived from the differential equation, and to obtain the two parts of the theorem simultaneously. To the second category belongs the consideration of

$$\int_0^1 \{P_n^m(x)\}^2\, dx,$$

when m and n are complex quantities; we find that the integral can be expressed as a higher hypergeometric series of argument unity, and we can determine all the cases in which this series reduces to the simple term just written down. In a similar manner we can evaluate integrals of the type

$$\int_0^1 \frac{P_n^m(x)\, x^k}{(1-x^2)^l}\, dx, \quad \int_0^1 \{P_n^m(x)\}^2\, (1-x^2)^{l_2}\, dx,$$

$$\int_0^1 P_n^m(x)\, Q_n^m(x)\, dx, \ \&c.$$

The results invariably lead to hypergeometric series of higher order. And not only can every special case hitherto discovered, which admits of expression in simple form, be obtained, but we are in a position to find other special cases in which a like phenomenon arises. The theory thus outlined links integrals connected with generalised Legendre functions to special types of hypergeometric series of higher order; it indicates the existence of a sequence of higher hypergeometric functions, with special properties, and shows that various fundamental integrals, involving generalised Legendre functions in the subject of integration, furnish for general complex values of m and n simple types of such higher functions.

The theory of some of the most important expansions connected with generalised Legendre functions will be considered in Part IV.; the investigations are of a somewhat different nature and lead up to Appell's hypergeometric series of two variables.

§ 58. *The difference equation for the generalised Legendre functions.*

We will first show that $P_n^m(x)$ and $Q_n^m(x)$ satisfy the difference equation

$$(2n+1)\,xf(n)-(n-m+1)f(n+1)-(n+m)f(n-1)=0.$$

This equation for the case when $m=0$, and for the function $P_n(x)$, is ascribed by Heine* to Gauss. The completely general form appears to be due to Hobson† [(125), p. 522].

We have seen in § 21 that, when $|\arg(-x^2)|<\pi$,

$$Q_n^m(x)=K\frac{1}{(2x)^{n+1}}\int\frac{\Gamma(2s+n+m+1)\,\Gamma(-s)}{\Gamma(n+\frac{3}{2}+s)}\left(-\frac{1}{4x^2}\right)^s ds,$$

where K denotes the expression

$$-\frac{\sqrt{\pi}}{2\pi\iota}\,\frac{\sin(n+m)\,\pi}{\sin n\pi}\left(\frac{x^2-1}{x^2}\right)^{\frac{1}{2}m}.$$

The contour of the integral is parallel to the imaginary axis apart from suitable loops, if necessary, to ensure that the sequence 0, 1, 2, ... lies to the right, and the sequence $-\frac{1}{2}(n+m+1)$, $-\frac{1}{2}(n+m+2)$, ... to the left of the contour. We may obviously, in general, so choose the loops that we pass over no poles of the subject of integration when we move the contour a distance unity parallel to the real axis keeping its shape unchanged.

Therefore

$$\begin{aligned}(n-m+1)\,Q^m{}_{n+1}(x)+(n+m)\,Q^m{}_{n-1}(x)&\\ =K\frac{1}{(2x)^n}\int&[(n-m+1)\,s+(n+m)\,(s+n+\tfrac{1}{2})]\\ &\times\frac{\Gamma(2s+n+m)\,\Gamma(-s)}{\Gamma(s+n+\frac{3}{2})}\left(-\frac{1}{4x^2}\right)^s ds\\ =K\frac{(2n+1)\,x}{(2x)^{n+1}}\int&\frac{\Gamma(2s+n+m+1)\,\Gamma(-s)}{\Gamma(s+n+\frac{3}{2})}\left(-\frac{1}{4x^2}\right)^s ds\\ =(2n+1)\,x\;&Q_n^m(x).\end{aligned}$$

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* Heine, *loc. cit.*, Bd. I., pp. 91, 92. The reader may also consult pp. 95–97 and a note of p. 197 of this treatise.

† Cf., however, F. Neumann, *Beiträge*, Leipzig (1878), p. 74.

Thus $Q_n^m(x)$ satisfies the given difference relation when $|\arg(-x^2)| < \pi$.

Since $Q_n^m(x)$ is an analytic function on the dissected x-plane, the given relation is true in general.

Again, since (§ 12)

$$P_n^m(x) = \frac{\tan n\pi}{\pi}\{Q_n^m(x) - Q^m{}_{-n-1}(x)\},$$

the same difference equation is valid for $P_n^m(x)$.

The difference equation just found is, in many ways, complementary to the differential equation for the generalised Legendre functions with which we started in § 1. It may be made fundamental in the theory of the functions. For a general theory of the linear difference equation of the second order with linear coefficients reference may be made to a paper by the author,* where the general solution of such an equation is given, and references to the literature of the subject will be found.

§ 59. *A generalisation of Rodrigues' formula.*

For general complex values of n, if m be a positive integer and x does not lie on the cross cut $(-\infty, 1)$, we have

$$P_n^{m-n}(x) = \frac{(x^2-1)^{\frac{1}{2}(m-n)}}{2^n\Gamma(n+1)}\frac{d^m}{dx^m}(x^2-1)^n.$$

And for similar values of m and n, when $-1 < x < 1$,

$$(-)^m P_n^{m-n}(x) = \frac{(1-x^2)^{\frac{1}{2}(m-n)}}{2^n\Gamma(n+1)}\frac{d^m}{dx^m}(1-x^2)^n.$$

By the author's investigation of the Binomial Theorem† we see that, if $|\arg(x \pm 1)| < \pi$,

$$-\frac{1}{2\pi\iota}\int\Gamma(s-n)\,\Gamma(-s)\frac{(x-1)^s}{2^s}\,ds = \Gamma(-n)\{\tfrac{1}{2}(x+1)\}^n,$$

and therefore

$$-\frac{1}{2\pi\iota}\int\Gamma(s-n)\,\Gamma(-s)\,2^{-s}(x-1)^{s+n}\,ds = 2^{-n}\Gamma(-n)(x^2-1)^n.$$

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* Barnes, *Messenger of Mathematics* (1904), Vol. XXXIV., pp. 52–71. The particular equation, when $m = 0$, is discussed in § 10 of the paper cited.

† *Loc. cit.*, § 15 (p. 117).

In this formula

$$(x^2-1)^n=(x-1)^n(x+1)^n,$$

where $$|\arg(x\pm1)|<\pi,$$

and the contour of the integral is drawn parallel to the imaginary axis with loops, if necessary, to ensure that the positive sequence of poles of the subject of integration lies to the right, and the negative sequence to the left of the contour.

On differentiation we see that, if m be a positive integer,

$$-\frac{1}{2\pi\iota}\int\frac{\Gamma(s-n)\,\Gamma(-s)\,\Gamma(s+n+1)}{2^s\Gamma(s+n+1-m)}(x-1)^{s+n-m}\,ds$$

$$=2^{-n}\Gamma(-n)\frac{d^m}{dx^m}(x^2-1)^n.$$

Thus, by § 6,

$$P_n^{m-n}(x)=\frac{1}{\Gamma(1+n-m)}\left(\frac{x+1}{x-1}\right)^{\frac{1}{2}(m-n)}$$

$$\times F\{-n,\ n+1;\ 1+n-m;\ \tfrac{1}{2}(1-x)\}$$

$$=\frac{(x^2-1)^{\frac{1}{2}(m-n)}}{2^n\Gamma(n+1)}\frac{d^m}{dx^m}(x^2-1)^n,$$

the first of the given results.

If in this result we put $x+\epsilon\iota$ for x, where $-1<x<1$, we have, in the limit when ϵ vanishes,

$$P_n^{m-n}(x+\epsilon\iota)=e^{-(m-n)\frac{1}{2}\pi\iota}P_n^{m-n}(x),\quad\text{by § 13,}$$

and $$\{(x+\epsilon\iota)^2-1\}^{\frac{1}{2}(n-m)}=\{1-(x+\epsilon\iota)^2\}^{\frac{1}{2}(n-m)}e^{(n-m)\frac{1}{2}\pi\iota};$$

and therefore

$$(-)^m P_n^{m-n}(x)=\frac{(1-x^2)^{\frac{1}{2}(m-n)}}{2^n\Gamma(n+1)}\frac{d^m}{dx^m}(1-x^2)^n,$$

the second of the given results.

If in either of the previous results we make $m=n$, we see that, when n is a positive integer, we have Rodrigues' formula*

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

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* For the history of this formula the reader may consult Heine, *loc. cit.*, p. 20.

The evaluation of $\int_0^1 P_n^m(x)\, x^k\, dx$ *for general complex values of* m, n, *and* k, *such that* $R(m) < 2$, $R(k) > -1$, *and also when* m *is a non-negative integer and* $R(k) > -1$.

§ 60. Let C be a simple contour passing in a positive direction from the origin round the point 1 and back to the origin. Then, if $R(k) > -1$, we may show that for general complex values of m, n, and k

$$\frac{1}{\pi\iota}\int_C P_n^m(x)\, x^k\, dx = \frac{2^m \Gamma\{\frac{1}{2}(k+1)\}}{\Gamma(1-m)\,\Gamma\{\frac{1}{2}m\}\,\Gamma\{\frac{1}{2}(k+3-m)\}}$$

$$\times {}_3F_2\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n),\ 1-\tfrac{1}{2}m\ ;\ 1-m,\ \tfrac{1}{2}(k+3-m)\ ;\ 1\},$$

where ${}_3F_2\{\alpha_1, \alpha_2, \alpha_3\ ;\ \beta_1, \beta_2\ ;\ x\}$, is a higher hypergeometric function defined, when $|x| < 1$, by the equality

$${}_3F_2\{\alpha_1, \alpha_2, \alpha_3\ ;\ \beta_1, \beta_2\ ;\ x\}$$

$$= \frac{\Gamma(\beta_1)\,\Gamma(\beta_2)}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)\,\Gamma(\alpha_3)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+n)\,\Gamma(\alpha_2+n)\,\Gamma(\alpha_3+n)}{\Gamma(\beta_1+n)\,\Gamma(\beta_2+n)\,n!}\, x^n.$$

Some properties of such series have been considered in a recent memoir by the author, where references to the literature of the subject may be found.* It is easily seen that the series given in the above equality is convergent if $R(k) > -2$.

Let L be a contour which closely embraces the positive half of the real axis, and has indentations to ensure, if necessary, that points belonging to negative sequences of poles of the subject of integration are not embraced by the contour. Then, if $(x^2-1)^s = (x-1)^s (x+1)^s$, where $|\arg(x \pm 1)| < \pi$, we have by the result of § 18

$$P_n^m(x)\, 2^{-m}\Gamma\{-\tfrac{1}{2}(m+n)\}\,\Gamma\{\tfrac{1}{2}(n+1-m)\}$$

$$= -\frac{1}{2\pi\iota}\int_L \frac{\Gamma\{s+\frac{1}{2}(n+1-m)\}\,\Gamma\{s-\frac{1}{2}(m+n)\}\,\Gamma(-s)}{\Gamma(1-m+s)}\,(x^2-1)^{s-\frac{1}{2}m}\, ds,$$

when $|x^2 - 1| < 1$.

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* *Proceedings of the London Mathematical Society*, Ser. 2, Vol. v., pp. 59–116.

Hence

$$2^{-m}\Gamma\{-\tfrac{1}{2}(m+n)\}\,\Gamma\{\tfrac{1}{2}(n+1-m)\}\int_C P_n^m(x)\,x^k\,dx$$

$$=-\frac{1}{2\pi\iota}\int_L \frac{\Gamma\{s+\tfrac{1}{2}(n+1-m)\}\,\Gamma\{s-\tfrac{1}{2}(m+n)\}\,\Gamma(-s)}{\Gamma(1-m+s)}\,ds$$

$$\times\int_C (x^2-1)^{s-\frac{1}{2}m}\,x^k\,dx.$$

The inversion of the order of integration is readily justified by splitting up the contour L into two parts respectively within and without a circle of large radius R, say L_1 and L_2. By the result of § 20 the original integral is convergent, if $R(k)>-1$.

Also, omitting for brevity the subject of integration in the double integral

$$\int_C P_n^m(x)\,x^k\,dx=\int_C\int_{L_1}+\int_C\int_{L_2}.$$

Now $\int_C\int_{L_1}=\int_{L_1}\int_C$, provided $R(k)>-1$, for both contours are finite, and the only possible singularity of the subject of integration is at $x=0$.

Again, if $|x^2-1|\leq 1$, and, if R is greater than the real part of $\frac{1}{2}m$, we have, by the asymptotic formula for the gamma function,

$$|\int_{L_2}|\leq\int_{L_2}\frac{|(x^2-1)^{R-\frac{1}{2}m}|}{|s^{\frac{3}{2}}|}\,|J|\,|ds|$$

$$\leq\frac{|(x^2-1)^{R-\frac{1}{2}m}|\,|J_1|}{R^{\frac{3}{2}}},$$

where $|J|$ and $|J_1|$ are finite positive quantities independent of x.

Hence $|\int_C\int_{L_2}|\leq\frac{|J_1|}{R^{\frac{1}{2}}}\int_C|(x^2-1)^{R-\frac{1}{2}m}|\,|x^k|\,|dx|$,

and this expression can be made as small as we please by sufficiently increasing R, if $R(k)>-1$.

Finally, $\int_{L_1}\int_C$ tends to a definite finite limit as R increases indefinitely, and the inversion is therefore justified.

Now, by Pochhammer's theory, for general complex values of s and m, if $R(k)>-1$,

$$\int_C(x^2-1)^{s-\frac{1}{2}m}x^k\,dx=\pi\iota\frac{\Gamma\{\frac{1}{2}(k+1)\}}{\Gamma(\frac{1}{2}m-s)\,\Gamma\{s+\frac{1}{2}(k+3-m)\}}.$$

We therefore have, if $R(k)>-1$,

$$\int_C P_n^m(x)\,x^k\,dx=\frac{\pi\iota\Gamma\{\frac{1}{2}(k+1)\}\,2^m}{\Gamma\{\frac{1}{2}(-m-n)\}\,\Gamma\{\frac{1}{2}(n+1-m)\}}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int_L\frac{\Gamma\{s+\frac{1}{2}(n+1-m)\}\,\Gamma\{s-\frac{1}{2}(m+n)\}\,\Gamma(-s)\,ds}{\Gamma(1-m+s)\,\Gamma(\frac{1}{2}m-s)\,\Gamma\{s+\frac{1}{2}(k+3-m)\}}$$

$$=\frac{\pi\iota\Gamma\{\frac{1}{2}(k+1)\}\,2^m}{\Gamma(1-m)\,\Gamma(\frac{1}{2}m)\,\Gamma\{\frac{1}{2}(k+3-m)\}}$$

$$\times{}_3F_2\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n),\ 1-\tfrac{1}{2}m;\ 1-m,\ \tfrac{1}{2}(k+3-m);\ 1\}.$$

We thus have the given theorem.

§ 61. *We may now show that, when $R(m)<2$, $R(k)>-1$, we have, for general complex values of m, n, and k,*

$$\int_0^1 P_n^m(x)\,x^k\,dx=2^{2m-1}\frac{\Gamma\{\frac{1}{2}(k+1)\}\sqrt{\pi}}{\Gamma\{\frac{1}{2}(1-m)\}\,\Gamma\{\frac{1}{2}(k+3-m)\}}$$

$$\times{}_3F_2\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n),\ 1-\tfrac{1}{2}m;\ 1-m,\ \tfrac{1}{2}(k+3-m);\ 1\};$$

and that if m be a positive integer ≥ 0, $R(k)>-1$, we have, for general complex values of n and k,

$$\int_0^1 P_n^m(x)\,x^k\,dx$$

$$=(-)^m\frac{\sqrt{\pi}\Gamma\{\frac{1}{2}(k+1)\}\,\Gamma(1+m+n)}{2^{2m+1}\Gamma\{\frac{1}{2}(1+m)\}\,\Gamma\{\frac{1}{2}(k+3+m)\}\,\Gamma(1-m+n)}$$

$$\times{}_3F_2\{\tfrac{1}{2}(n+1+m),\ \tfrac{1}{2}(m-n),\ 1+\tfrac{1}{2}m;\ 1+m,\ \tfrac{1}{2}(k+3+m);\ 1\}.$$

If $R(m)<2$, the integral to be considered is finite near $x=1$, for, by § 18, $P_n^m(x)\,(1-x^2)^{\frac{1}{2}m}$ is finite near $x=1$. Similarly, if $R(k)>-1$, it is finite near $x=0$.

Also, by § 13,

$$\int_C P_n^m(x)\,x^k\,dx=\{e^{\frac{1}{2}m\pi\iota}-e^{-\frac{1}{2}m\pi\iota}\}\int_0^1 P_n^m(x)\,x^k\,dx$$

$$=2\iota\sin(\tfrac{1}{2}\pi m)\int_0^1 P_n^m(x)\,x^k\,dx.$$

If then $R(m)<2$, $R(k)>-1$, we have by the preceding paragraph the first of the given results.

Again, when m is a positive integer or zero, we see by (3), § 13, that $P_n^m(x)\,(x^2-1)^{-\frac{1}{2}m}$ is finite near $x=1$. And therefore the integral $\int_1^1 P_n^m(x)\,x^k\,dx$ exists if $R(k)>-1$.

Now by the result just proved we have, on changing m into $-m$,

$$\int_0^1 P_n^{-m}(x)\, x^k\, dx = 2^{-2m-1} \frac{\Gamma\{\frac{1}{2}(k+1)\}\sqrt{\pi}}{\Gamma\{\frac{1}{2}(1+m)\}\,\Gamma\{\frac{1}{2}(k+3+m)\}}$$

$$\times {}_3F_2\{\tfrac{1}{2}(n+1+m),\ \tfrac{1}{2}(m-n),\ 1+\tfrac{1}{2}m;\ 1+m,\ \tfrac{1}{2}(k+3+m);\ 1\}.$$

Also, by § 13, when $-1<x<1$, and m is a positive integer or zero,

$$P_n^{-m}(x) = (-)^m P_n^m(x) \frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

Combining these two identities we have the second of the given results.

§ 62. *We can at once show as a corollary that for general values of* n *and* k, if $R(k)>-1$,

$$\int_0^1 P_n(x)\, x^k\, dx = 2^n \frac{\Gamma(k+1)}{\Gamma(k+n+2)} \frac{\Gamma\{\frac{1}{2}(k+n)+1\}}{\Gamma\{\frac{1}{2}(k-n)+1\}}.$$

For, if we make $m=0$ in either of the preceding formulæ, we get

$$\int_0^1 P_n(x) x^k dx = \frac{\Gamma\{\frac{1}{2}(k+1)}{2\Gamma\{\frac{1}{2}(k+3)\}} F\{\tfrac{1}{2}(n+1),\ -\tfrac{1}{2}n;\ \tfrac{1}{2}(k+3);\ 1\};$$

and this by Gauss' theorem is equal to

$$\frac{\Gamma\{\frac{1}{2}(k+1)\}\,\Gamma(\frac{1}{2}k+1)}{2\Gamma\{\frac{1}{2}(k-n)+1\}\,\Gamma\{\frac{1}{2}(k+3+n)\}},$$

which, in turn, by the multiplication theorem for the gamma function, is equal to the given result.

The result is, of course, well known for the special case when n and k are positive integers. References to the work of Legendre, Cayley, and Dirichlet will be found in Heine's treatise.*

When $k=0$, we have for general complex values of n

$$\int_0^1 P_n(x)\, dx = \frac{\sqrt{\pi}}{2\Gamma(1-\frac{1}{2}n)\,\Gamma\{\frac{1}{2}(n+3)\}}.$$

Other cases when the general integral $\int_0^1 P_n(x)\, x^k\, dx$ reduces to a product of gamma functions are readily given by examining the relations between k, m, and n, which must hold good in order that the function ${}_3F_2$ of § 61 may reduce to an ordinary Gaussian hypergeometric series.

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* Heine, *loc. cit.*, pp. 71–74.

§ 63. We can now show that, if $R(k)>-1$, $R(l+\frac{1}{2}m)<1$, we have, for general complex values of l, m, n, and k,

$$\int_0^1 \frac{P_n^m(x)\,x^k}{(1-x^2)^l}\,dx=\frac{2^{m-1}\Gamma(1-l-\frac{1}{2}m)\,\Gamma\{\frac{1}{2}(k+1)\}}{\Gamma(1-m)\,\Gamma\{\frac{1}{2}(k+3-m)-l\}}$$

$$\times {}_3F_2\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n),\ 1-l-\tfrac{1}{2}m;\ 1-m,\ \tfrac{1}{2}(k+3-m)-l;\ 1\};$$

and that, if m is a positive integer or zero, $R(k)>-1$, $R(l-\frac{1}{2}m)<1$, we have, for general complex values of n, l, and k,

$$\int_0^1 \frac{P_n^m(x)\,x^k}{(1-x^2)^l}\,dx=(-)^m\frac{\Gamma(1+m+n)\,\Gamma(1-l+\frac{1}{2}m)\,\Gamma\{\frac{1}{2}(k+1)\}}{2^{m+1}\Gamma(1-m+n)\Gamma(1+m)\Gamma\{\frac{1}{2}(k+3+m)-l\}}$$

$$\times {}_3F_2\{\tfrac{1}{2}(n+1+m),\ \tfrac{1}{2}(m-n),\ 1-l+\tfrac{1}{2}m;\ 1+m,\ \tfrac{1}{2}(k+3+m)-l;\ 1\}.$$

The theorem is an obvious extension of the result of § 61 and is proved by the same method and argument.

We first show that, if $R(k)>-1$, for general values of l, m, n, and k,

$$\frac{1}{\pi\iota}\int_C \frac{P_n^m(x)\,x^k}{(x^2-1)^l}\,dx=\frac{2^m\Gamma\{\frac{1}{2}(k+1)\}}{\Gamma(1-m)\,\Gamma\{\frac{1}{2}m+l\}\,\Gamma\{\frac{1}{2}(k+3-m)-l\}}$$

$$\times {}_3F_2\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n),\ 1-l-\tfrac{1}{2}m;\ 1-m,\ \tfrac{1}{2}(k+3-m)-l;\ 1\}.$$

From this we obtain at once the first relation. The second relation is deduced by means of the identity, valid when m is a positive integer or zero and $-1<x<1$,

$$P_n^{-m}(x)=(-)^m P_n^m(x)\,\frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

Special cases in which the series ${}_3F_2$ can be reduced to an ordinary hypergeometric series which may be expressed in terms of gamma functions by Gauss' formula may be easily obtained.

Two such cases are perhaps worthy of mention.

In the first of the given formulæ make $l=\frac{1}{2}m$. Then we see that, if $R(m)<1$, $R(k)>-1$, we have, for general complex values of m, n, and k,

$$\int_0^1 \frac{P_n^m(x)\,x^k}{(1-x^2)^{\frac{1}{2}m}}\,dx=\frac{\Gamma\{\frac{1}{2}(k+1)\}\,2^{m-1}}{\Gamma\{\frac{1}{2}(k+3)-m\}}$$

$$\times F\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(-m-n);\ \tfrac{1}{2}(k+3)-m;\ 1\}$$

$$=\frac{2^{m-1}\Gamma\{\frac{1}{2}(k+1)\}\,\Gamma(\frac{1}{2}k+1)}{\Gamma\{\frac{1}{2}k+1-\frac{1}{2}(n+m)\}\,\Gamma\{\frac{1}{2}(k+3-m+n)\}}.$$

Again, in the second of the given formulæ, make $l=-\frac{1}{2}m$. Then we see that, if m be a positive integer or zero, and if

$R(k)>-1$, we have, for general complex values of n and k,

$$\int_0^1 (1-x^2)^{\frac{1}{2}m} x^k P_n^m(x)\,dx$$

$$= \frac{(-)^m}{2^{m+1}} \frac{\Gamma\{\frac{1}{2}(k+1)\}\,\Gamma(\frac{1}{2}k+1)}{\Gamma\{1+\frac{1}{2}(k+m-n)\}\,\Gamma\{\frac{1}{2}(3+k+m+n)\}} \frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}.$$

The evaluation of integrals of the type

$$\int_{-1}^1 \{P_n^m(x)\}^2 (1-x^2)^k\,dx.$$

§ 64. The most important proposition in the elementary theory of Legendre functions is probably that expressed by the equalities

$$\int_{-1}^1 P_n^m(x)\,P^m{}_{n_1}(x)\,dx = 0 \quad (n \neq n_1)$$

and

$$= \frac{2\Gamma(1+m+n)}{\Gamma(1+n-m)\,(2n+1)} \quad (n=n_1),$$

the quantities m, n, n_1 being positive integers (zero included). The theorem can be substantially carried back to Laplace and Legendre; the usual proof depends upon the conjunction of Rodrigues' formula with the method of integration by parts.*

We proceed to obtain the result by applying Cauchy's theory of residues to the contour integrals introduced in the present theory.

When m and n are positive integers ($m \gtrless n$) and $-1<x<1$, we have, by § 13 (p. 113),

$$P_n^m(x) = (-)^m (1-x^2)^{\frac{1}{2}m} \frac{\Gamma(m+n+1)}{2^m \Gamma(1+n-m)\,\Gamma(m+1)}$$

$$\times F\{m-n,\ m+n+1;\ m+1;\ \tfrac{1}{2}(1-x)\}$$

$$= (-)^{m-1} \frac{(1-x^2)^{\frac{1}{2}m}}{2^m} \frac{1}{2\pi\iota} \int \frac{\Gamma(m+n+1+s)\Gamma(-s)}{\Gamma(1+n-m-s)\Gamma(1+m+s)} \{\tfrac{1}{2}(1-x)\}^s ds,$$

the contour being a finite oval including the points 0, 1, 2, ..., $n-m$, which are the sole poles of the subject of integration.

Hence

$$\int_{-1}^1 P^m{}_n(x)\,P_n^m(x)\,dx$$

$$= \left(\frac{1}{2\pi\iota}\right)^2 \iint \frac{\Gamma(m+n+1+s)\,\Gamma(-s)\,\Gamma(m+n_1+1+s_1)\,\Gamma(-s_1)}{\Gamma(1+n-m-s)\Gamma(1+m+s)\,\Gamma(1+n_1-m-s_1)\,\Gamma(1+m+s_1)}$$

$$\times \int_{-1}^1 \{\tfrac{1}{2}(1-x)\}^{s+s_1+m} \{\tfrac{1}{2}(1+x)\}^m dx\,ds\,ds_1.$$

* For references the reader may consult Heine, *loc. cit.*, Bd. I., pp. 67–70 and pp. 251–255.

The contour for s and s_1 are finite, and can be so chosen that upon them $R(s+s_1)+m>-1$. Also $m\geq 0$. Hence the inversion in the order of integration is justifiable, and we have

$$\frac{1}{2\,\Gamma(m+1)}\int_{-1}^{1} P_n^m(x)\,P_{n_1}^m(x)\,dx$$

$$=\left(\frac{1}{2\pi\iota}\right)^2\iint\frac{\Gamma(s+n)\,\Gamma(s_1+n_1)}{\Gamma(s)\,\Gamma(s_1)}\;\frac{\Gamma(m+1-s)\,\Gamma(m+1-s_1)}{\Gamma(2+n-s)\,\Gamma(2+n_1-s_1)}$$

$$\times\frac{\Gamma(s+s_1-m-1)}{\Gamma(s+s_1)}\,ds\,ds_1,$$

where the contours include among the poles of the subject of integration the points

$$s=m+1,\ m+2,\ \ldots,\ n+1,\ s_1=m+1,\ m+2,\ \ldots,\ n_1+1,$$

and no others.

Suppose now that we write the integral, as is evidently allowable, in the form

$$\frac{1}{2\pi\iota}\int\frac{\Gamma(s_1+n_1)\,\Gamma(m+1-s_1)}{\Gamma(s_1)\,\Gamma(2+n_1-s_1)}\,ds_1$$

$$\times\frac{1}{2\pi\iota}\int\frac{\Gamma(s+n)\,\Gamma(m+1-s)\,\Gamma(s+s_1-m-1)}{\Gamma(s)\,\Gamma(2+n-s)\,\Gamma(s+s_1)}\,ds.$$

For any assigned value of s_1 of finite modulus the second integral vanishes when taken round an infinitely large contour, for the modulus of the subject of integration behaves when $|s|$ is large like $1/|s|^2$.

The last expression is therefore equal to

$$-\frac{1}{2\pi\iota}\sum_{r=0}^{m}\int\frac{\Gamma(s_1+n_1)\,\Gamma(m+1-s_1)}{\Gamma(s_1)\,\Gamma(2+n_1-s_1)}$$

$$\times\frac{\Gamma(s_1+r)\,\Gamma(m+n+1-r-s_1)\,(-)^r}{\Gamma(m+1-r-s_1)\,\Gamma(s_1+r+1+n-m)\,\Gamma(m+1-r)\,r!}\,ds_1$$

$$=-\frac{1}{2\pi\iota}\sum_{r=0}^{m}\frac{(-)^r}{r!\,(m-r)!}\int\frac{\Gamma(s+m+1+n_1)\,\Gamma(s+m+1+r)}{\Gamma(s+m+1)\,\Gamma(s+r+2+n)}$$

$$\times\frac{\Gamma(-s)}{\Gamma(1+n_1-m-s)}\;\frac{\Gamma(n-r-s)}{\Gamma(-r-s)}\,ds,$$

the contour being a finite oval including the positive sequences of poles of the subject of integration.

Now the subject of integration is a quotient of two algebraic polynomials in s. The only possible poles within the contour are at $s=0,\ 1,\ \ldots,\ n_1-m$, and at these points $1+r+s-n$ is negative or zero if $n_1<n$.

Thus no pole exists within the contour of $n_1<n$, and under the same proviso

$$\int_{-1}^{1} P_n^m(x)\, P^m_{n_1}(x)\, dx = 0.$$

In a similar manner we may show that the integral is zero, if $n_1>n$.

When $n_1=n$, there is one pole within the final contour when $r=m$, $s=n_1-m$. We therefore have, when n and m are positive integers or zero,

$$\frac{1}{2\Gamma(m+1)}\int_{-1}^{1}\{P_n^m(x)\}^2\,dx$$

$$= \frac{\Gamma(2n+1)\,\Gamma(n+m+1)}{\Gamma(2n+2)}\ \frac{1}{\Gamma(n-m+1)\,m!},$$

or
$$\int_{-1}^{1}\{P_n^m(x)\}^2\,dx = \frac{2\Gamma(n+m+1)}{(2n+1)\,\Gamma(n-m+1)}.$$

We therefore have the two required results.

§ 65. It is natural to enquire whether the result just obtained can be extended to complex values of m and n.

We proceed to show that for all values real or complex of m and n, except $n=-p-\frac{1}{2}$, where p is zero or a positive integer, we have the relation

$$\int_C \{P_n^m(x)\}^2\,dx = \frac{2\iota\sin\pi m\,\Gamma(1+m+n)}{(2n+1)\,\Gamma(1-m+n)}$$

$$-\frac{\iota\sin\pi m}{\Gamma(-m-n)\,\Gamma(1-m+n)}\sum_{t=0}^{\infty}\frac{\Gamma(-m+n+1+t)\,\Gamma(m+n+1+t)}{\Gamma(t+1)\Gamma(2n+2+t)(t+n+\frac{1}{2})},$$

where C is the contour of § 60 *which passes in a positive direction from the origin round the point* 1 *and back again to the origin.* The infinite series which intervenes in this formula, and which disappears when m and n are positive

integers, is a particular type of the higher hypergeometric series ${}_3F_2\{\alpha_1, \alpha_2, \alpha_3;\ \rho_1, \rho_2;\ x\}$, for which $x=1$ and $\alpha_3=\rho_2-1$.

We shall make use of a theorem which may be expressed by the formula

$$[F\{\alpha, \beta;\ \gamma;\ x\}]^2={}_3F_2[2\alpha, 2\beta, \gamma-\tfrac{1}{2};\ 2\gamma-1, \gamma;\ x\},$$

when $\alpha+\beta+\frac{1}{2}=\gamma$.

This theorem arises from the consideration of the differential equation satisfied by the square of the solution of Legendre's equation, and has its origin in the investigations of Clausen*, F. Neumann†, Appell‡, and Goursat§. We shall refer to it as Goursat's theorem, and in Part IV. of the present paper it will be established as a particular case of a new and more general formula in the theory of Appell's hypergeometric functions of two variables.

By § 18, if $R(x)>0$, $|1-x^2|<1$, we have

$$P_n^m(x)=\frac{2^m(x^2-1)^{-\frac{1}{2}m}}{\Gamma(1-m)}F\{\tfrac{1}{2}(n+1-m), \tfrac{1}{2}(-m-n);\ 1-m;\ 1-x^2\}.$$

Hence

$$\{P_n^m(x)\}^2=\frac{2^{2m}(x^2-1)^{-m}}{\{\Gamma(1-m)\}^2}$$

$$\times {}_3F_2\{n+1-m, -m-n, \tfrac{1}{2}-m;\ 1-2m, 1-m;\ 1-x^2\}$$

$$=\frac{(x^2-1)^{-m}}{\sqrt{(\pi)}\,\Gamma(n+1-m)\,\Gamma(-m-n)}\left(-\frac{1}{2\pi\iota}\right)$$

$$\times\int_L\frac{\Gamma(-s)\Gamma(n+1-m+s)\Gamma(-m-n+s)\Gamma(\frac{1}{2}-m+s)}{\Gamma(1-2m+s)\Gamma(1-m+s)}(x^2-1)^s ds.$$

When $|1-x^2|<1$, we may take the contour L to embrace closely the positive half of the real axis, there being indentations, if necessary, to exclude points belonging to negative sequences of poles of the subject of integration.

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* Clausen, *Crelle*, Bd. 3 (1828), pp. 89–95.

† F. Neumann, *Beiträge, Leipzig* (1878), pp. 1–156, especially p. 81, et seq.

‡ Appell, *Annales de l'Ecole Normale Supérieure* (1881), Séries 2, T. 10, pp. 391–423; and *Liouville*, Séries 3, T. 10, pp. 407–428, especially pp. 420–421.

§ Goursat, *Annales de l'Ecole Normale Supérieure*, Series 2, T. 12 (1883), pp. 261–286 and pp. 395–430. The formula is given on p. 284 of Goursat's paper: though due to Clausen, it may well be associatad with Goursat's name on acconnt of his elaborate discussion of the questions which its existence raises. Some amount of the previous investigations is given in two supplementary notes in Heine's second volume, pp. 357–360 and pp. 361–373.

We therefore have, as in § 60,

$$\int_C \{P_n^m(x)\}^2 dx = -\frac{1}{2\pi\iota\sqrt{(\pi)}\,\Gamma(n+1-m)\,\Gamma(-m-n)}$$

$$\times \int \frac{\Gamma(-s)\Gamma(n+1-m+s)\Gamma(-m-n+s)\Gamma(\frac{1}{2}-m+s)}{\Gamma(1-2m+s)\Gamma(1-m+s)} \int_C (x^2-1)^{t-m} dx$$

$$= \frac{-2^{2m}\Gamma(\frac{1}{2})\,\Gamma(1-2m)}{2\pi\,\Gamma(n+1-m)\,\Gamma(-m-n)\,\Gamma(\frac{1}{2}-m)\,\Gamma(1-m)}$$

$$\times \int_L \frac{\Gamma(-s)\Gamma(n+1-m+s)\Gamma(-m-n+s)\Gamma(\frac{1}{2}-m+s)\sin\pi(m-s)}{\Gamma(1-2m+s)\Gamma(s-m+\frac{3}{2})} ds$$

$$= \frac{2\iota\sin\pi m}{\Gamma(2-2m)}\, {}_3F_2\{n+1-m, -m-n, \tfrac{1}{2}-m;\ 1-2m, \tfrac{3}{2}-m;\ 1\}.$$

It is evident that the last series is always convergent, except for special values of m which give rise to infinite terms in the denominators of the series.

The result just obtained may be evidently written in the form

$$\int_C \{P_n^m(x)\}^2 dx = \iota\sin\pi m\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}$$

$$\times \left(-\frac{1}{2\pi\iota}\right) \int \frac{\Gamma(-s)\,\Gamma(n+1-m+s)}{\Gamma(1-2m+s)\,\Gamma(1+m+n-s)\,(\frac{1}{2}-m+s)} dx.$$

The contour includes the positive sequences of poles of the subject of integration, but excludes the negative sequence and the point $m-\frac{1}{2}$.

When $|s|$ is very large, the modolus of the subject of integration is of order $1/|s|^2$. We may therefore employ the usual process of bending the contour of the integral backwards (cf. § 6), and we get, if n is not equal to $-p-\frac{1}{2}$,

$$\iota\sin\pi m\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}$$

$$\times \left[\sum_{t=0}^{\infty} \frac{\Gamma(t+n+1-m)\,(-)^{t-1}}{t!\,\Gamma(-m-n-t)\,\Gamma(2+2n+t)\,(t+n+\frac{1}{2})} + \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{3}{2}+n)}\right],$$

the isolated term arising from the pole $s=m-\frac{1}{2}$.

This result evidently yields the theorem stated at the beginning of the present paragraph.

§ 66. We proceed to consider certain deductions from the preceding theorem.

In the first place we see that, if $m+n$ is a positive integer or zero, we have, for all real or complex values of m and n, except $n=-p-\frac{1}{2}$,

$$\int_C \{P_n^m(x)\}^2 dx = \frac{2\iota \sin \pi m}{2n+1} \frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}.$$

Again, if $R(m)<1$, we have

$$\int_C \{P_n^m(x)\}^2 dx = 2\iota \sin \pi m \int_0^1 \{P_n^m(x)\}^2 dx.$$

Hence, for all real or complex values of m and n (except $n=-p-\frac{1}{2}$) such that $R(m)<1$, we have

$$\int_0^1 \{P_n^m(x)\}^2 dx = \frac{\Gamma(1+m+n)}{(2n+1)\Gamma(1-m+n)}$$

$$- \frac{1}{2\Gamma(-m-n)\Gamma(1-m+n)} \sum_{t=0}^{\infty} \frac{\Gamma(n+1-m+t)\Gamma(1+m+n+t)}{t!\Gamma(2+2n+t)(t+n+\frac{1}{2})}.$$

When $n \neq -p-\frac{1}{2}$, $R(m)<1$, and $m+n$ is a positive integer or zero, we have, for all values of m and n,

$$\int_0^1 \{P_n^m(x)\}^2 dx = \frac{\Gamma(1+m+n)}{(2n+1)\Gamma(1-m+n)}.$$

We may readily prove that this result is also valid when m and n are positive integers ($n \geq m \geq 0$). In this case, by § 13, the integral exists, and when $0 \leq x < 1$, by § 13,

$$P_n^{-m}(x) = (-)^m P_n^m(x) \frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

Changing the sign of m in the result just established, and, using this relation, we have the required result. We have thus re-established the later half of the theorem of § 64.

A reduction formula for $\int_C \{P_n^m(x)\}^2\,dx$.

We can now show that, for general complex values of m and n,

$$(2n+1)\,\Gamma(n+1-m)\,\Gamma(-m-n)\int_C\{P_n^m(x)\}^2\,dx$$
$$+ \text{ the same expression with } (n-1) \text{ written for } n,$$
$$= \frac{4n\iota\sin\pi m}{m^2-n^2}.$$

For, by § 65, if the contour closely embrace the positive half of the real axis and exclude all poles of the subject of integration except the sequence 0, 1, 2, ..., we have

$$\Gamma(n+1-m)\,\Gamma(-m-n)\int_C\{P_n^m(x)\}^2\,dx$$
$$= -\frac{\sin\pi m}{2\pi}\int\frac{\Gamma(-s)\,\Gamma(n+1-m+s)\,\Gamma(-m-n+s)\,e^{\pi\iota s}}{\Gamma(1-2m+s)\,(\tfrac{1}{2}-m+s)}\,ds.$$

Hence the expression to be considered

$$= -\frac{\sin\pi m}{2\pi}\int\frac{\Gamma(-s)\,\Gamma(n-m+s)\,\Gamma(-m-n+s)\,e^{\pi\iota s}}{\Gamma(1-2m+s)}\,\Theta ds,$$

where

$$\Theta = \frac{(2n+1)\,(s-m+n)+(2n-1)\,(s-m-n)}{s-m+\tfrac{1}{2}} = 4n.$$

Thus the expression under consideration

$$= 4n\iota\sin\pi m\,\frac{\Gamma(n-m)\,\Gamma(-n-m)}{\Gamma(1-2m)}\,F\{n-m,\,-n-m;\;1-2m;\;1\}$$
$$= \frac{4n\iota\sin\pi m}{m^2-n^2}.$$

We thus have the given result.

When $R(m)<1$, we see that, for general complex values of n,

$$(2n+1)\,\Gamma(n+1-m)\,\Gamma(-n-m)\int_0^1\{P_n^m(x)\}^2\,dx$$
$$+ \text{ the same expression with } (n-1) \text{ written for } n,$$
$$= \frac{2n}{m^2-n^2}.$$

In particular, when $m=0$, we have, for general complex values of n,

$$(2n+1)\int_0^1\{P_n(x)\}^2\,dx-(2n-1)\int_0^1\{P_{n-1}(x)\}^2\,dx=\frac{2\sin\pi n}{\pi n}.$$

Thus, when n is a positive integer, we have Legendre's result

$$(2n+1)\int_0^1\{P_n(x)\}^2\,dx=1.$$

§ 67. *The exceptional case when* $n=-p-\frac{1}{2}$.

We now proceed to consider the particular case when $n=-p-\frac{1}{2}$, which was excluded from the investigations of § 65 and the first part of § 66.

When $n=-\frac{1}{2}$, we have, as in § 65,

$$\int_C \{P_{-\frac{1}{2}}^m(x)\}^2 dx = \iota \sin \pi m \frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2}-m)}$$

$$\times \left(-\frac{1}{2\pi\iota}\right) \int \frac{\Gamma(-s)\,\Gamma(\frac{1}{2}-m+s)}{\Gamma(1-2m+s)\,\Gamma(\frac{1}{2}+m-s)\,(\frac{1}{2}-m+s)}\, ds,$$

where the contour of the integral includes the negative sequence of poles $m-\frac{1}{2}-t$, $t=0, 1, 2, \ldots$, together with the point $(m-\frac{1}{2})$.

Now the residue at $\epsilon=0$ of

$$\Gamma(\epsilon)\,\Gamma(\tfrac{1}{2}-m-\epsilon)/\{\Gamma(\tfrac{1}{2}-m+\epsilon)\,\Gamma(1-\epsilon)\,\epsilon\}$$

is

$$-2\gamma-2\psi(\tfrac{1}{2}-m),$$

where γ is Euler's constant and $\psi(x)=\dfrac{d}{dx}\log\Gamma(x)$.

We therefore have, for general complex values of m,

$$\int_C \{P_{-\frac{1}{2}}^m(x)\}^2\, dx = -\iota \sin \pi m \frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2}-m)} \left[2\gamma + 2\psi(\tfrac{1}{2}-m)\right.$$

$$\left. + \frac{1}{\Gamma(\frac{1}{2}-m)\,\Gamma(\frac{1}{2}+m)} \sum_{t=1}^{\infty} \frac{\Gamma(\frac{1}{2}-m+t)\,\Gamma(\frac{1}{2}+m+t)}{t\{\Gamma(t+1)\}^2}\right].$$

If $R(m)<1$, we have

$$\int_0^1 \{P_{-\frac{1}{2}}^m(x)\}^2\, dx = -\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2}-m)}\{\gamma+\psi(\tfrac{1}{2}-m)\}$$

$$-\frac{1}{2\{\Gamma(\frac{1}{2}-m)\}^2} \sum_{t=1}^{\infty} \frac{\Gamma(\frac{1}{2}-m+t)\,\Gamma(\frac{1}{2}+m+t)}{t\{\Gamma(t+1)\}^2}.$$

Similarly, if m be zero or a positive integer,

$$\int_0^1 \{P_{-\frac{1}{2}}^m(x)\}^2 = -\frac{\Gamma(\frac{1}{2}+m)}{\Gamma(\frac{1}{2}-m)}\{\gamma+\psi(\tfrac{1}{2}+m)\}$$

$$-\frac{1}{2\{\Gamma(\frac{1}{2}-m)\}^2} \sum_{t=1}^{\infty} \frac{\Gamma(\frac{1}{2}+m+t)\,\Gamma(\frac{1}{2}-m+t)}{t\{\Gamma(t+1)\}^2}.$$

These results can equally be deduced as limiting cases of those obtained in §§ 65 and 66.

When $n=-p-\frac{1}{2}$ and p is an integer >0, the reader may easily show in a similar manner that, for general complex values of m,

$$\int_C \{P_n^m(x)\}^2\,dx = -\frac{2\iota\sin\pi m}{2n+1}\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}$$

$$-\frac{\iota\sin\pi m}{\Gamma(-m-n)\,\Gamma(1-m+n)}\sum_{t=0}^{\infty}\frac{\Gamma(t-m-n)\,\Gamma(t+m-n)}{\Gamma(t-2n)\,\Gamma(t+1)\,(t-n-\frac{1}{2})}.$$

This, again, may be deduced as a limiting case of the result of § 65.

§ 68. An obvious extension of the procedure of § 65 shows us that, for general complex values of m, n, and k,

$$\int_C \{P_n^m(x)\}^2\,(x^2-1)^k\,dx = \frac{2^{2m}\pi\iota\,\Gamma(\frac{1}{2})}{\{\Gamma(1-m)\}^2\,\Gamma(m-k)\,\Gamma(k-m+\frac{3}{2})}$$

$$\times {}_4F_3\{n+1-m,\ -m-n,\ \tfrac{1}{2}-m,\ 1+k-m;\ 1-2m,\ 1-m,\ \tfrac{3}{2}+k-m;\ 1\}.$$

The hypergeometric series is convergent, except for special values of m and $k-m$ for which factors of various terms in the denominator become zero.

When $k=-1,\ 0,\ -m,\ n-\frac{1}{2},\ -n-\frac{3}{2}$, etc., the hypergeometric series reduces to one of lower order.

Let us take as a special example the case when $k=-1$.

We find

$$\int_C \frac{\{P_n^m(x)\}^2}{x^2-1}\,dx = \frac{2^{2m}\pi\iota\,\Gamma(\frac{1}{2})}{\{\Gamma(1-m)\}^2\,\Gamma(1+m)\,\Gamma(\frac{1}{2}-m)}$$

$$\times {}_3F_2\{n+1-m,\ -m-n,\ -m;\ 1-m,\ 1-2m;\ 1\}$$

$$= -\iota\sin\pi m\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(-s)\,\Gamma(s+n+1-m)}{\Gamma(1+m+n-s)\,\Gamma(1-2m+s)\,(s-m)}\,ds,\ \text{if } m\neq 0.$$

The contour includes the positive sequence of poles of the subject of integration, but excludes the negative sequence and the point $s=m$.

When $|s|$ is large the subject of integration behaves like $\frac{1}{|s|^2}$. We may therefore bend the contour backwards and obtain

$$\int_C \frac{\{P_n^m(x)\}^2}{x^2-1}\,dx = \frac{\iota \sin \pi m}{m}\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}$$

$$+\frac{\iota \sin m\pi}{\Gamma(1-m+n)\,\Gamma(-m-n)}\sum_{t=0}^{\infty}\frac{\Gamma(n+1-m+t)\,\Gamma(n+1+m+t)}{t!\,\Gamma(t+2n+2)\,(t+n+1)}.$$

This result is valid for general complex values of m and n. The exceptional cases can be easily enumerated and considered. The most important occurs when $m=0$. In this case we have

$$\frac{1}{\pi\iota}\int_C \frac{\{P_n(x)\}^2}{x^2-1}\,dx = 1.$$

When $R(m)<0$, we have

$$\int_C \frac{\{P_n^m(x)\}^2}{1-x^2}\,dx = 2\iota \sin \pi m \int_0^1 \frac{\{P_n^m(x)\}^2}{1-x^2}\,dx.$$

Thus, if $R(m)<0$, for general complex values of m and n, we have

$$\int_0^1 \frac{\{P_n^m(x)\}^2}{1-x^2}\,dx = -\frac{1}{2m}\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)} - \frac{1}{2\Gamma(1-m+n)\,\Gamma(-m-n)}$$

$$\times \sum_{t=0}^{\infty}\frac{\Gamma(n+1-m+t)\,\Gamma(n+1+m+t)}{t!\,\Gamma(2+2n+t)\,(t+n+1)}.$$

In particular, if $R(m)<0$, and if $n+m$ be a positive integer or zero, we have, for all values of m and n,

$$\int_0^1 \frac{\{P_n^m(x)\}^2}{1-x^2}\,dx = -\frac{1}{2m}\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)}.$$

(An apparent exception occurs when $n+1-m$ is a negative integer, but this is excluded by the given conditions.)

From this last result we may readily deduce that, if n and m are positive integers ($n \geq m > 0$), we have

$$\int_0^1 \frac{\{P_n^m(x)\}^2 dx}{1-x^2} = \frac{1}{2m}\,\frac{\Gamma(1+m+n)}{\Gamma(1-m+n)} \quad \ldots\ldots\ldots (1).$$

By what has just been proved, we have, under the given conditions,

$$\int_0^1 \frac{\{P_\nu^{-m}(x)\}^2}{1-x^2}\,dx = \frac{1}{2m}\,\frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

But in this case, by § 13,

$$P_n^{-m}(x) = (-)^m P_n^m(x)\,\frac{\Gamma(1-m+n)}{\Gamma(1+m+n)}.$$

We therefore have the given result.

Other special cases can be investigated in the same way. We have discussed the special case in question as the result (1) is given by Heine*.

The evaluation of integrals of the type

$$\int\{Q_n^m(x)\}^2\,dx,\ \int Q_n^m(x)\,P_n^m(x)\,dx.$$

§ 69. *We now proceed to show that, for general real or complex values of m and n,*

$$\frac{1}{\pi\iota}\int_C\{Q_n^m(x)\}^2\,dx = \frac{\sin^2(n+m)\,\pi\cos n\pi}{\sin^2 n\pi}\,\frac{\Gamma(n+1+m)}{\Gamma(n+1-m)} \times \sum_{t=0}^{\infty}\frac{\Gamma(m+n+1+t)\,\Gamma(n+1-m+t)}{t!\,\Gamma(2n+2+t)\,(2t+2n+1)},$$

where C is the contour of § 60 *which passes in a positive direction from the origin round the point* 1 *and back again to the origin.*

By § 21, we have, when $|1-x^2|>1$,

$$Q_n^m(x) = \frac{-\pi}{\sin n\pi}\,\frac{\Gamma(\frac{1}{2})}{2^{n+1}\Gamma(-m-n)\,\Gamma(n+\frac{3}{2})}\,(x^2-1)^{-\frac{1}{2}(m+1)} \times F\{\tfrac{1}{2}(n+m+1),\ \tfrac{1}{2}(n+1-m);\ n+\tfrac{3}{2};\ 1/(1-x^2)\}.$$

Hence, by Goursat's theorem quoted in § 65,

$$\{Q_n^m(x)\}^2 = \frac{\pi^3}{\sin^2 n\pi}\,\frac{(x^2-1)^{-m-1}}{2^{2n+2}\{\Gamma(-m-n)\,\Gamma(n+\frac{3}{2})\}^2} \times {}_3F_2\{n+m+1,\ n-m+1,\ n+1;\ 2n+2,\ n+\tfrac{3}{2};\ 1/(1-x^2)\}$$

$$= \tfrac{1}{2}(\sqrt{\pi})\,\frac{\sin^2(m+n)\,\pi}{\sin^2 n\pi}\,\frac{\Gamma(1+n+m)}{\Gamma(1+n-m)} \times \left(-\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(m+s)\Gamma(-m+s)\Gamma(s)\Gamma(n+1-s)}{\Gamma(n+1+s)\Gamma(\frac{1}{2}+s)}\,(x^2-1)^{-s}\,ds \ldots (A).$$

* Heine, *loc. cit.*, Bd. 1., 47 (a), p. 253.

The final expression is valid, provided $|\arg(x^2-1)|<\pi$, if the contour of the integral be taken parallel to the imaginary axis, with suitable indentations, if necessary, to ensure that the positive sequence of poles of the subject of integration lies to the right of the contour and negative sequences to the left.

If $|R(m)|<1$ we can so draw the contour of the integral that $R(s)<1$. And then

$$\int_C (x^2-1)^{-s}\,ds = 2\iota \sin \pi s \int_0^1 (1-x^2)^{-s}\,ds = \frac{\pi\iota\,\Gamma(\frac{1}{2})}{\Gamma(s)\,\Gamma(\frac{3}{2}-s)}.$$

Thus, under this limitation,

$$\frac{1}{\pi\iota}\int_C \{Q_n^m(x)\}^2\,dx = \tfrac{1}{2}\pi \frac{\sin^2(m+n)\,\pi}{\sin^2 n\pi} \frac{\Gamma(1+n+m)}{\Gamma(1+n-m)}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(s+m)\,\Gamma(s-m)\,\Gamma(n+1-s)}{\Gamma(n+1+s)\,\Gamma(\frac{1}{2}+s)\,\Gamma(\frac{3}{2}-s)}\,ds.$$

Now the final integral is convergent for all values of m since, when $|s|$ is large, the subject of integration behaves like $1/|s|^2$. Further, by § 22, the integral on the left-hand side of the equality is convergent for all values of m. Hence, since the functions concerned are analytic functions of m, we may dispense with the condition $|R(m)|<1$.

Using Cauchy's theory of residues we obtain at once the given result.

§ 70. *We may next show that, if* $|R(m)|<1$, $R(n)>-\frac{1}{2}$,

$$\frac{\Gamma(1+n-m)}{\Gamma(1+n+m)}\int_1^\infty \{Q_n^m(x)\}^2\,dx = \tfrac{1}{4}\pi \frac{\sin^2(m+n)\,\pi}{\sin^2 n\pi}$$

$$\times\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(s-m)\,\Gamma(s+m)\,\Gamma(n+1-s)}{\sin \pi s\,\Gamma(n+1+s)\,(s-\frac{1}{2})}\,ds,$$

where the contour of the final integral separates the poles $1, 2, 3, \ldots$; $n+1, n+2, \ldots$ from $\frac{1}{2}$; $0, -1, -2, \ldots$; $\pm m, \pm m-1, \ldots$.

We have

$$\int_1^\infty (x^2-1)^{-s}\,dx = \frac{\Gamma(1-s)\,\Gamma(s-\frac{1}{2})}{2\Gamma(\frac{1}{2})},$$

if $\frac{1}{2}<R(s)<1$.

Take the formula (A) of § 69. The contour of the integral in this formula can be drawn to satisfy the condition $\frac{1}{2} < R(s) < 1$, if $R(n) > -\frac{1}{2}$, $|R(m)| < 1$.

Now, at $x = 1$, $\{Q_n^m(x)\}^2$ behaves like $(1-x)^{\pm m}$, by § 9.

and at $x = \infty$, „ „ „ x^{-2n-2}, by § 21.

We thus have the given theorem.

If we apply Cauchy's theorem to the given contour integral we obtain the value of $\int_1^\infty \{Q_n^m(x)\}^2 dx$ when $|R(m)| < 1$, $R(n) > -\frac{1}{2}$. The result which involves the sum of two particular hypergeometric functions of the type ${}_3F_2$ is somewhat complicated. A reduction formula can, however, be obtained which is comparatively simple.

§ 71. *We proceed to show that, if* $|R(m)| < 1$, $R(n) > -\frac{1}{2}$, *we have, for general complex values of* m *and* n, *the reduction formula*

$$(2n+1)\frac{\Gamma(n+1-m)}{\Gamma(n+1+m)}\int_1^\infty \{Q_n^m(x)\}^2 dx,$$

minus the same expression with $(n-1)$ *written for* n

$$= -\frac{\sin^2(m+n)\pi}{\sin^2 n\pi}\,\frac{\Gamma(1-m)\,\Gamma(1+m)}{n^2-m^2}.$$

By the result of § 70 the expression on the left-hand side in this equality

$$= \tfrac{1}{4}\pi\frac{\sin^2(m+n)\pi}{\sin^2 n\pi}\left(-\frac{1}{2\pi\iota}\right)\int \frac{\Gamma(s-m)\,\Gamma(s+m)\,\Gamma(n-s)}{\sin \pi s\,\Gamma(n+1+s)} \times \left\{\frac{(n-s)(2n+1)-(n+s)(2n-1)}{s-\frac{1}{2}}\right\} ds$$

$$= -\pi n\frac{\sin^2(m+n)\pi}{\sin^2 n\pi} I,$$

where

$$I = -\frac{1}{2\pi\iota}\int \frac{\Gamma(s-m)\,\Gamma(s+m)\,\Gamma(n-s)}{\sin \pi s\,\Gamma(n+1+s)}\,ds,$$

and the contour of the integral can be taken to embrace solely the sequences of poles $n, n+1, n+2, \ldots$ and $1, 2, 3, \ldots$.

By Cauchy's theorem

$$I=\sum_{t=0}^{\infty}\frac{\Gamma(t+n-m)\,\Gamma(t+n+m)}{\sin\pi n\,\Gamma(2n+1+t)\,t!}$$

$$+\sum_{t=1}^{\infty}\frac{(-)^{t-1}\,\Gamma(t-m)\,\Gamma(t+m)\,\Gamma(n-t)}{\pi\,\Gamma(n+1+t)}$$

$$=\frac{1}{\sin\pi n\,(n^2-m^2)}-\frac{\Gamma(1+m)\,\Gamma(1-m)}{\sin\pi n\,\Gamma(2+n)\,\Gamma(2-n)}$$

$$\times\,{}_2\mathfrak{F}_2\{1-m,\,1+m;\;2+n,\,2-n;\;1\},$$

where

$${}_2\mathfrak{F}_2\{\alpha,\,\beta;\;\gamma,\,\delta;\;x\}=1+\frac{\alpha\beta}{\gamma\delta}\,x+\frac{\alpha\,(\alpha+1)\,\beta\,(\beta+1)}{\gamma\,(\gamma+1)\,\delta\,(\delta+1)}\,x^2+\ldots.$$

The latter series is convergent when $|x|<1$, or when $|x|=1$ and $R(\gamma+\delta-\alpha-\beta)>1$. Now, by a theory due to M. J. M. Hill* and the author conjointly,

$${}_2\mathfrak{F}_2\{1-m,\,1+m;\;2-n,\,2+n;\;1\}$$

$$=(1+n)\,{}_2\mathfrak{F}_2\{1-n+m,\,1-m-n;\;2-n,\,2;\;1\}$$

$$=\frac{1-n^2}{n^2-m^2}\,[F\{m-n,\,-m-n;\;1-n;\;1\}-1].$$

Hence

$${}_2\mathfrak{F}_2\{1-m,\,1+m;\;2-n,\,2+n;\;1\}$$

$$=\frac{n^2-1}{n^2-m^2}+\frac{1-n^2}{n^2-m^2}\,\frac{\Gamma(1+n)\,\Gamma(1-n)}{\Gamma(1+m)\,\Gamma(1-m)}.$$

And therefore $I=\dfrac{\Gamma(1-m)\,\Gamma(1+m)}{\pi n\,(n^2-m^2)}$.

We thus have the given theorem.

§ 72. As particular cases of the preceding theorem we may obtain results due to Hargreaves.†

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* Hill, *Proceedings of the London Mathematical Society*, Series 2, Vol. v., pp. 335–341. The extension of Hill's theorem was communicated by the author to the same Society in September 1907, and was followed by a different proof of this extension due to Hill. *Ibid.*, p. xxi.

* Hargreaves, *Proceedings of the London Mathematical Society*, Series 1, Vol. xxix., p. 118.

When $m=0$, we have for general complex values of n such that $R(n)>-\frac{1}{2}$

$$(2n+1)\int_1^\infty \{Q_n(x)\}^2\,dx-(2n-1)\int_1^\infty \{Q_{n-1}(x)\}^2\,dx=-1/n^2,$$

a result obtained by Hargreaves when n is real.

Again, by § 70, we have

$$\int_1^\infty \{Q_0(x)\}^3\,dx=\tfrac{1}{4}\pi\left(-\frac{1}{2\pi\iota}\right)\int\frac{ds}{(\sin\pi s)^2\,s\,(s-\frac{1}{2})},$$

where the contour includes solely the poles $s=1, 2, 3, \ldots$.

Thus, by Cauchy's theory of residues,

$$\int_1^\infty \{Q_0(x)\}^2\,dx=\tfrac{1}{4}\sum_{t=1}^{\infty}\frac{1}{t\,(t-\frac{1}{2})}\left\{\frac{1}{t}+\frac{1}{t-\frac{1}{2}}\right\}$$

$$=2\left\{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\right\}=\tfrac{1}{6}\pi^2.$$

Combining this with the previous result, we see that, when n is a positive integer,

$$\int_1^\infty \{Q_n(x)\}^2\,dx=\frac{1}{(2n+1)}\sum_{t=0}^{\infty}\frac{1}{(n+1+t)^2}$$

$$=\frac{\psi^{(2)}(n+1)}{2n+1},$$

where $\psi^{(2)}(z)=\dfrac{d^2}{dz^2}\log\Gamma(z)$.

§ 73. *A contour integral for* $P_n^m(x)\,Q_n^m(x)$.

We may now show that, if $|\arg(x^2-1)|<\pi$,

$$P_n^m(x)\,Q_n^m(x)=\frac{\Gamma(\frac{1}{2})}{2\sin\pi n\,\Gamma(n+1-m)\,\Gamma(-m-n)}$$

$$\times\left(\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(s-m)\,\Gamma(s)\,\Gamma(\frac{1}{2}-s)\,\Gamma(1+n-s)}{\Gamma(1-s-m)\,\Gamma(s+n+1)}\,(x^2-1)^{-s}\,ds,$$

the contour of the integral being parallel to the imaginary axis with the usual indentations, if necessary.

By (A), § 69, we have, if the contour of the integral apart from indentations be parallel to the imaginary axis, and if

$|\arg(x^2-1)|<\pi$,

$$\{Q_n^m(x)\}^2=\frac{\pi^{\frac{3}{2}}\sin(m+n)\pi}{2\sin^2 n\pi\Gamma(1+n-m)\Gamma(-n-m)}$$

$$\times\left(\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(s-m)\Gamma(s+m)\Gamma(s)\Gamma(n+1-s)}{\Gamma(\frac{1}{2}+s)\Gamma(n+1+s)}(x^2-1)^{-s}ds.$$

Therefore

$$\{Q_n^m(x)\}^2-\{Q^m{}_{-n-1}(x)\}^2=-\frac{\Gamma(\frac{1}{2})}{2\sin^2 n\pi\Gamma(1+n-m)\Gamma(-n-m)}$$

$$\times\left(\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(s-m)\Gamma(s+m)\Gamma(s)\Gamma(n+1-s)\Gamma(-n-s)}{\Gamma(\frac{1}{2}+s)}\Theta(x^2-1)^{-s}ds,$$

where

$$\Theta=\sin(m+n)\pi\sin(n+s)\pi+\sin(m-n)\pi\sin(n-s)\pi$$

$$=\sin(m+s)\pi\sin 2n\pi.$$

Hence

$$\{Q_n^m(x)\}^2-\{Q^m{}_{-n-1}(x)\}^2=\frac{\pi^{\frac{3}{2}}\cos n\pi}{\sin n\pi\Gamma(1+n-m)\Gamma(-n-m)}$$

$$\times\left(\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(s-m)\Gamma(s)\Gamma(\frac{1}{2}-s)\Gamma(n+1-s)}{\Gamma(1-m-s)\Gamma(1+n+s)}\frac{\cos\pi s}{\sin\pi(n+s)}(x^2-1)^{-s}ds.$$

Also, by § 65, we readily see that, under similar limitations,

$$\pi^2\cot^2 n\pi\{P_n^m(x)\}^2=\frac{\pi^{\frac{3}{2}}\cos^2 n\pi}{\sin^2 n\pi\Gamma(1+n-m)\Gamma(-n-m)}$$

$$\times\left(\frac{1}{2\pi\iota}\right)\int\frac{\Gamma(s-m)\Gamma(s)\Gamma(\frac{1}{2}-s)\Gamma(n+1-s)}{\Gamma(1-m-s)\Gamma(1+n+s)}\frac{\sin\pi s}{\sin\pi(n+s)}(x^2-1)^{-s}ds.$$

Now, by § 12,

$$\{Q_n^m(x)\}^2-\{Q^m{}_{-n-1}(x)\}^2+\pi^2\cot^2 n\pi\{P_n^m(x)\}^2$$

$$=2\pi\cot n\pi P_n^m(x)\,Q_n^m(x).$$

Combining the last three results, we have the given theorem. It is evident that this theorem enables us at once to express $P_n^m(x)Q_n^m(x)$ as the sum of two series of the type ${}_3F_2$ when $|x^2-1|$ is either greater than or less than unity.

In the latter case we have the additional restriction $|\arg(x^2-1)|<\pi$.

It may be remarked that the same result may be obtained directly by taking the expressions for $P_n^m(x)$ and $Q_n^m(x)$ given in §§ 19 and 21 respectively. These are

$$P_n^m(x)=\frac{2^m\Gamma(n+\frac{1}{2})}{\Gamma\{\frac{1}{2}(n+1-m)\}\,\Gamma\{1+\frac{1}{2}(n-m)\}}(x^2-1)^{\frac{1}{2}n}$$
$$\times F\{\tfrac{1}{2}(m-n),\ \tfrac{1}{2}(-m-n);\ \tfrac{1}{2}-n;\ 1/(1-x^2)\}$$
$$+\frac{2^m\Gamma(-n-\frac{1}{2})}{\Gamma\{\frac{1}{2}(-m-n)\}\,\Gamma\{\frac{1}{2}(1-n-m)\}}(x^2-1)^{-\frac{1}{2}(n+1)}$$
$$\times F\{\tfrac{1}{2}(n+1-m),\ \tfrac{1}{2}(n+1+m);\ \tfrac{3}{2}+n;\ 1/(1-x^2)\},$$

and

$$Q_n^m(x)=-\frac{\pi^{\frac{1}{2}}}{2^{n+1}\sin n\pi\,\Gamma(-m-n)\,\Gamma(n+\frac{3}{2})}(x^2-1)^{-\frac{1}{2}(n+1)}$$
$$\times F\{\tfrac{1}{2}(n+m+1),\ \tfrac{1}{2}(n+1-m);\ \tfrac{3}{2}+n;\ 1/(1-x^2)\}.$$

We multiply these two expressions together and use the formulæ

$$\{F(\alpha,\beta;\ \gamma;\ z)\}^2=F\{2\alpha,\ 2\beta,\ \gamma-\tfrac{1}{2};\ 2\gamma-1,\ \gamma;\ z\},$$
$$F\{\alpha,\beta;\ \gamma;\ z\}\,F\{\tfrac{1}{2}-\alpha,\ \tfrac{1}{2}-\beta;\ 2-\gamma;\ z\}$$
$$=F\{\alpha-\beta+\tfrac{1}{2},\ \beta-\alpha+\tfrac{1}{2},\ \tfrac{1}{2};\ 2-\gamma,\ \gamma;\ z\},$$

where $\gamma=\alpha+\beta+\frac{1}{2}$, $2\alpha=n+1-m$, $2\beta=n+1+m$.

The first of these formulæ has been already quoted in § 65 as Goursat's formula. The second can be obtained in a similar manner, and is also due to Clausen and Goursat.*

§ 74. The connection of this theory with that of linear differential equations will be evident to the reader.

The general solution of the equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left\{n(n+1)-\frac{m^2}{1-x^2}\right\}y=0$$

is $$AP_n^m(x)+BQ_n^m(x),$$

where A and B are arbitrary constants.

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* Goursat, *loc. cit.*, p. 284.

Hence the general solution of the equation

$$(1-x^2)\frac{d^3y}{dx^3}-6x\frac{d^2y}{dx^2}$$

$$+\left\{\frac{6x^2-4m^2-2}{(1-x^2)}+4n(n+1)\right\}\frac{dy}{dx}-\frac{4n(n+1)x}{1-x^2}y=0\ldots(B)$$

is $$A\{P_n^m(x)\}^2+BP_n^m(x)\,Q_n^m(x)+C\{Q_n^m(x)\}^2,$$

where A, B, and C are arbitrary constants.

And we have just proved that three linearly independent solutions of the latter equation are

$$\int\frac{\Gamma(s-m)\,\Gamma(\tfrac{1}{2}-s)\,\Gamma(n+1-s)\,\Gamma(-n-s)}{\Gamma(1-m-s)\,\Gamma(1-s)}(x^2-1)^{-s}\,ds,$$

$$\int\frac{\Gamma(s-m)\,\Gamma(m+s)\,\Gamma(s)\,\Gamma(n+1-s)}{\Gamma(n+1+s)\,\Gamma(\tfrac{1}{2}+s)}(x^2-1)^{-s}\,ds,$$

and $$\int\frac{\Gamma(s-m)\,\Gamma(\tfrac{1}{2}-s)\,\Gamma(s)\,\Gamma(n+1-s)}{\Gamma(1-m-s)\,\Gamma(n+1+s)}(x^2-1)^{-s}\,ds.$$

A large number of other integrals can be derived from this fundamental set. The equation (B) is substantially (when the necessary transformations are made) a particular kind of higher hypergeometric equation, whose solutions admit a general theory of transformation analogous to the transformations of generalised Legendre functions. We are at the beginning of a theory of particular types of higher hypergeometric functions of apparently indefinite range. The development of such a theory would evidently carry us far from the main object of the present paper.

§ 75. The contour integral just obtained for $P_n^m(x)\,Q_n^m(x)$ can evidently be used to evaluate integrals of the type $\int P_n^m(x)\,Q_n^m(x)\,x^k(x^2-1)^l\,dx$ between the limits 0 and 1 or 1 and ∞, when such integrals exist. The existence of such integrals is determined by the consideration that

$P_n^m(x)\,Q_n^m(x)$ behaves at $x=\infty$ like x^{-1} or x^{-2n-2},

" " $x=0$ " unity,

" " $x=1$ " $(x-1)^{-m}$ or unity.

In order to compress the algebra to which such integrals necessarily lead we will merely consider the integral

$$\int_0^1 P_n^m(x)\, Q_n^m(x)\, dx,$$

which exists, if $R(m) < 1$. *We will show that, under this limitation, for general complex values of m and n,*

$$\int_0^1 P_n^m\, Q_n^m(x)\, dx = \frac{\pi}{4 \sin n\pi\, \Gamma(n+1-m)\, \Gamma(-n-m)}$$

$$\times \frac{1}{2\pi\iota} \int \frac{\Gamma(s-m)\,\Gamma(s)\,\Gamma(1-s)\,\Gamma(1+n-s)}{\Gamma(1-s-m)\,\Gamma(s+n+1)\,(\frac{1}{2}-s)} \cos(\tfrac{1}{2}m-s)\,\pi\, ds,$$

where the contour of the integral may be taken to embrace the positive half of the real axis, and to include solely the positive sequences of poles of the subject of integration and the point $s = \frac{1}{2}$.

When $-1 < x < 1$, we have, by §§ 13 and 14, in the limit when $\epsilon = 0$,

$$P_n^m(x)\, Q_n^m(x) = \tfrac{1}{2} P_n^m(x) \{ Q_n^m(x+\epsilon\iota) + Q_n^m(x-\epsilon\iota) \}$$

$$= \tfrac{1}{2} \{ P_n^m(x+\epsilon\iota)\, Q_n^m(x+\epsilon\iota)\, e^{\frac{1}{2}m\pi\iota} + P_n^m(x-\epsilon\iota)\, Q_n^m(x-\epsilon\iota)\, e^{-\frac{1}{2}m\pi\iota} \}.$$

Now, by § 73, we see that, if $0 < x < 1$, in the limit when $\epsilon = 0$,

$$P_n^m(x \pm \epsilon\iota)\, Q_n^m(x \pm \epsilon\iota) = \frac{\Gamma(\frac{1}{2})}{2 \sin \pi n\, \Gamma(n+1-m)\, \Gamma(-n-m)}$$

$$\times \frac{1}{2\pi\iota} \int \frac{\Gamma(s-m)\,\Gamma(s)\,\Gamma(\frac{1}{2}-s)\,\Gamma(1+n-s)}{\Gamma(1-s-m)\,\Gamma(s+n+1)} e^{\mp\pi\iota s} (1-x^2)^{-s}\, ds,$$

the upper or lower signs being taken throughout.

The integral is convergent if its contour, apart from any indentations which may be necessary, is taken parallel to the imaginary axis. For, when $s = u + \iota v$ and $|v|$ is large, the subject of integration is at most of order $1/|s|^{\frac{3}{2}}$.

We therefore have, when $0 < x < 1$,

$$P_n^m(x)\, Q_n^m(x) = \frac{\Gamma(\frac{1}{2})}{2 \sin \pi n \Gamma(n+1-m)\, \Gamma(-n-m)}$$

$$\times \frac{1}{2\pi\iota} \int \frac{\Gamma(s-m)\,\Gamma(s)\,\Gamma(\frac{1}{2}-s)\,\Gamma(1+n-s)}{\Gamma(1-s-m)\,\Gamma(s+n+1)} \cos(\tfrac{1}{2}m-s)\pi (1-x^2)^{-s} ds.$$

Hence, by the usual type of argument, we have the given result.

By evaluating the contour integral we may express

$$\int_0^1 P_n^m(x)\, Q_n^m(x),$$

when $R(m)<1$, in terms of particular types of higher hypergeometric functions of argument unity. The result is complicated.

§ 76. It is, however, possible to obtain a comparatively simple reduction formula for such integrals. *We proceed to show that, if* $R(m)<1$,

$$\Gamma(n+1-m)\,\Gamma(-n-m)\sin\pi n\,(2n+1)\int_0^1 P_n^m(x)\,Q_n^m(x)\,dx$$

minus the same expression with $(n-1)$ *written in place of* n,

$$=\frac{\pi m\cos\frac{1}{2}(m\pi)}{m^2-n^2}+\frac{\pi n\cos(n+\frac{1}{2}m)\,\pi}{(m^2-n^2)}.$$

By § 75, the left-hand side in this equality

$$=\pi^2 n\,\frac{1}{2\pi\iota}\int\frac{\Gamma(s-m)\,\Gamma(n-s)\cos(\frac{1}{2}m-s)\,\pi}{\Gamma(1-s-m)\,\Gamma(n+1+s)\sin s\pi}\,ds,$$

where the contour includes solely the sequences of poles

$$s=n,\ n+1,\ n+2,\ \ldots \text{ and } s=1,\ 2,\ 3,\ \ldots.$$

By Cauchy's theory of residues we therefore have

$$-\frac{\pi n\cos(\frac{1}{2}m-n)\,\pi\sin\pi(n+m)}{\sin\pi n}\sum_{t=0}^{\infty}\frac{\Gamma(n-m+t)\,\Gamma(n+m+t)}{t!\,\Gamma(2n+1+t)}$$

$$+\frac{\pi n\cos\frac{1}{2}(m\pi)\sin\pi m}{\sin\pi n}\sum_{t=1}^{\infty}\frac{\Gamma(t-m)\,\Gamma(t+m)}{\Gamma(1+n+t)\,\Gamma(1-n+t)}$$

$$=-\pi n\,\frac{\cos(\frac{1}{2}m-n)\,\pi}{n^2-m^2}\,\frac{\sin\pi(n+m)}{\sin\pi n}$$

$$-\pi n\,\frac{\cos\frac{1}{2}(m\pi)}{m^2-n^2}\,\frac{\sin\pi m}{\sin\pi n}\left\{1-\frac{\Gamma(1-m)\,\Gamma(1+m)}{\pi n}\sin\pi n\right\},\ \text{by § 71,}$$

$$=\frac{\pi n}{(m^2-n^2)\sin\pi n}\{\cos(\tfrac{1}{2}m-n)\,\pi\sin\pi(n+m)-\cos\tfrac{1}{2}(m\pi)\sin\pi m\}$$

$$+\frac{\pi m}{m^2-n^2}\cos\tfrac{1}{2}(m\pi).$$

We thus have the given result.

§ 77. Various particular cases of the preceding formula are of interest.

When $m=0$ we see that we have, for general complex values of n,

$$(2n+1)\int_0^1 P_n(x)\,Q_n(x)\,dx-(2n-1)\int_0^1 P_{n-1}(x)\,Q_{n-1}(x)\,dx = (\cos\pi n)/n,$$

a somewhat elegant result.

When n is an integer this reduces to a formula of Hargreaves.*

In the particular case, when $m=n=0$, we have, by § 75,

$$\int_0^1 P_0(x)\,Q_0(x)\,dx=-\tfrac{1}{4}\left(\frac{1}{2\pi\iota}\right)\int\frac{\{\Gamma(s)\}^2\,\Gamma(1-s)}{\Gamma(1+s)\,(\tfrac{1}{2}-s)}\cos\pi s\,ds,$$

the contour including solely the positive sequence of poles of the subject of integration.

By Cauchy's theorem the integral is equal to

$$\tfrac{1}{2}\sum_{t=0}^{\infty}\frac{1}{(t+1)\,(2t+1)}=\sum_{t=0}^{\infty}\left\{\frac{1}{2t+1}-\frac{1}{2t+2}\right\}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$$

We therefore have, if n be a positive integer,

$$\int_0^1 P_n(x)\,Q_n(x)\,dx=\frac{(-)^n}{2n+1}\left\{\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+3}-\frac{1}{n+4}+\ldots\right\}$$

$$=\frac{(-)^n}{2\,(2n+1)}[\psi(\tfrac{1}{2}n+1)-\psi\{\tfrac{1}{2}(n+1)\}],$$

where $\psi(z)=\dfrac{d}{dz}\log\Gamma(z)$.

This is, apart from an obvious error, Hargreaves' result (*loc. cit.*, p. 120).

§ 78. It is evident that the theory discussed in the preceding paragraphs might be almost indefinitely extended. The methods used perhaps indicate sufficiently the procedure by which further results may be developed. The generalisations which have been given include, as particular cases, practically all of the results which have been previously obtained in the restricted part of the theory to which we have limited our consideration in the present Part of this paper.

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* Hargreaves, *loc. cit.* (11), p. 119.

(To be continued.)

A CLASS OF GROUPS IN AN ARBITRARY REALM CONNECTED WITH THE CONFIGURATION OF THE 27 LINES ON A CUBIC SURFACE.

[Second Paper.]

By Professor L. E. Dickson.

1. In the former paper with the same title, *Quarterly Journal*, Vol. xxxiii. (1901), pp. 145–173, the writer considered the linear homogeneous group $G(F)$, in a general field F, leaving invariant a cubic form in 27 variables with 45 terms:

$$(1) \qquad C = \sum_{i,j}^{1,\ldots,6} x_i y_j z_{ij} + \sum_{(15)} z_{12} z_{34} z_{56} \quad (z_{ij} = -z_{ji}),$$

the second sum comprising the 15 terms of the Pfaffian [123456]. The form C defines the configuration of the 45 triangles formed by the 27 straight lines on a general cubic surface. When F is the field of all real and complex numbers, $G(F)$ is the continuous group of 78 parameters forming one of Killing's five simple groups not occurring in Lie's four infinite systems of simple groups.

The present addition gives a decided simplification in certain of the proofs and in the formulæ for the inverse of the general transformation of the group. The correction of a minor error leads to a revised formula for the order of the group. At the outset is discussed the solution of a system of Pfaffian equations.

2. We employ the $\binom{m}{2}$ variables

$$\xi_{ij} = -\xi_{ji} \quad (i, j = 1, \ldots, m;\ i \neq j).$$

The Pfaffian of lowest order is

$$[ijkl] = \xi_{ij}\xi_{kl} + \xi_{ik}\xi_{lj} + \xi_{il}\xi_{jk}.$$

The equation $[ijkl] = 0$ is unaltered by any permutation of i, j, k, l.

Theorem. *If* $\xi_{ab} \neq 0$, *the* $\binom{m}{4}$ *distinct equations*

$$(2) \qquad [ijkl] = 0 \quad (i, j, k, l = 1, \ldots, m)$$

can be derived from the $\binom{m-2}{2}$ *equations involving* ζ_{ab}:

(3) $[abjk]=0$ $(j, k=1, \ldots, m;\ j, k$ *distinct from* $a, b)$.

Let r, s, t be any distinct subscripts no one of which is a or b. From the expansion of $[aabrst]\equiv 0$, we get

$$\zeta_{at}[abrs]-\zeta_{as}[abrt]+\zeta_{ar}[abst]=\zeta_{ab}[arst].$$

Hence, from (2), $[arst]=0$. Similarly, $[brst]=0$.

Let r, s, t, u be any distinct subscripts no one of which is a or b. Expanding the equal Pfaffians

$$[abrstu],\quad [tuabrs],$$

we find that $\zeta_{ab}[rstu]$ is a linear function of Pfaffians $[bstu]$, $[abrs]$, etc., each involving a or b, and hence is zero. Hence $[rstu]=0$, and the theorem follows.

A similar proof leads to the more general theorem:

For $\zeta_{ab}\neq 0$ *and* μ *even, the equations*

(4) $\qquad [i_1 i_2 \ldots i_\mu]=0 \quad (i_1, \ldots, i_\mu=1, \ldots, m)$

can be derived from $[abj_1 \ldots j_{\mu-2}]=0$.

3. Theorem. *In the Galois field of order* p^n, *the number of sets of solutions not all zero of the Pfaffian equations* (2) *is*

(5) $\qquad N_m=(p^{nm}-1)(p^{n(m-1)}-1)\div(p^{2n}-1)$.

If π_m denotes the number of sets of solutions with $\zeta_{m1}, \ldots, \zeta_{m\,m-1}$ not all zero, we have the recursion formula

(6) $\qquad N_m=N_{m-1}+\pi_m \quad (m>4)$.

For any given value $\neq 0$ of $\zeta_{m\,m-1}$, equations (3) with $a=m$, $b=m-1$ uniquely determine the ζ_{jk} $(j\leqq m-2,\ k\leqq m-2)$ in terms of ζ_{mi} and $\zeta_{m-1\,i}$ $(i=1, \ldots, m-2)$. Hence there are $(p^n-1)\,p^{n(2m-4)}$ sets of solutions with $\zeta_{m\,m-1}\neq 0$.

For $\zeta_{m\,m-1}=0$, and any given value $\neq 0$ of $\zeta_{m\,m-2}$, we may take ζ_{mi} $(i<m-2)$ and $\zeta_{m-2\,i}$ $(i<m,\ i\neq m-2)$ arbitrary, and uniquely determine the remaining ζ's by (3). Hence there are

$$(p^n-1)\,p^{n(2m-5)} \text{ sets with } \zeta_{m\,m-1}=0,\ \zeta_{m\,m-2}\neq 0.$$

The argument may be continued. Finally, there are

$$(p^n-1)\,p^{n(m-2)} \text{ sets with } \zeta_{mi}=0\ (i=m-1, \ldots, 2),\ \zeta_{m1}\neq 0,$$

since the ζ_{1i} $(1<i<m)$ may then be taken arbitrarily.

Hence, for $m > 4$, we have

$$\pi_m = (p^n - 1)(p^{n(2m-4)} + p^{n(2m-5)} + \ldots + p^{n(m-2)}) = (p^{n(m-1)} - 1)\, p^{n(m-2)}.$$

In view of (6), formula (5) now follows by simple induction from $m-1$ to m, it being true for $m = 4$.

4. The discussion on pp. 160—165 may be replaced by the following simpler treatment based on the normalization of the linear functions.

The functions (17) satisfying (18) are of three types:

First, let $\xi_2, \ldots, \xi_6$ be not all zero. A suitable product of the transformations A_{ikl} $(i > 1,\ k > 1)$ will replace $\sum_{i=2}^{6} \xi_i x_i$ by x_2, and hence replace function (17) by

$$\eta_1 y_1 + \xi_1 x_1 + x_2 + \Sigma \zeta_{ij} z_{ij} \quad (i, j = 2, \ldots, 6;\ i < j).$$

Conditions (18) thus reduce to

$$\eta_1 = [3456]_\zeta,\ \zeta_{2i} = 0 \quad (i = 3, 4, 5, 6).$$

Now ξ_1 and ζ_{ij} $(i, j = 3, \ldots, 6)$ remain arbitrary. Since $\xi_2, \ldots, \xi_6$ have any values not all zero in the $GF[p^n]$, the number of sets is

$$p^{7n}(p^{5n} - 1).$$

Next, let $\xi_2 = \ldots = \xi_6 = 0$, and let the ζ's be not all zero. Then conditions (18) reduce to $[ijkl] = 0$ $(i, j, k, l = 2, \ldots, 6)$. By § 3 above, the number of sets of solutions ζ_{ij}, not all zero, is

$$N_5 = (p^{5n} - 1)(p^{2n} + 1).$$

Since η_1 and ξ_1 are arbitrary, the number of sets is $p^{2n} N_5$.

Finally, let $\xi_i = \zeta_{ij} = 0$ $(i, j = 2, \ldots, 6)$. Then conditions (18) are satisfied, and (17) becomes $\phi = \eta_1 y_1 + \xi_1 x_1$. Under S, $x_1' = x_1,\ x_2' = \phi$. Hence $\eta_1 \neq 0$. The number of sets is $p^n (p^n - 1)$.

Hence the total number of functions (17) satisfying (18) is

$$p^{7n}(p^{5n} - 1) + p^{2n} N_5 + p^n (p^n - 1) = p^n (p^{8n} - 1)(p^{3n} + 1).$$

5. The complicated proof of the theorem on page 162 may be replaced by the following simple one. There are three cases:

First, let $\xi_1, \ldots, \xi_6$ be not all zero. Then there exists a product of the A's which replaces $\Sigma \xi_i x_i$ by x_1, each y by

a function of the y's, each z by a function of the x's. But there are p^{11n} functions (16) with $\xi_1 = 1$, $\xi_i = 0$ $(i > 1)$, by top of p. 152. Hence, for the present case, there are $p^{11n}(p^{6n} - 1)$ sets.

Next, let $\xi_i = 0$ $(i = 1, \ldots, 6)$, and let the η's be not all zero. Then the product MS replaces x_1 by

$$-\Sigma\eta_j x_j + \Sigma\zeta_{ij} z_{ij}.$$

As in the first case, the number of sets is $(p^{6n} + 1)N$, where N is the number of functions $x_1 + \Sigma\zeta_{ij} z_{ij}$. For the latter,

$$\xi_1 = 1, \ \xi_2 = \ldots = \xi_6 = 0, \ \eta_i = 0 \ \ (i = 1, \ \ldots, \ 6).$$

Hence conditions (12) and (13) reduce to

$$\zeta_{1i} = 0, \quad [klmn]_\zeta = 0 \quad (k, \ l, \ m, \ n = 2, \ \ldots, \ 6).$$

Thus (§ 3) $N = N_5 + 1$, the case $\zeta = 0$ being included.

Finally, let $\xi_i = \eta_i = 0$ $(i = 1, \ \ldots, \ 6)$. The number of sets is N_6 (§ 3).

The total number of sets is therefore $(p^{9n} - 1)(p^{8n} + p^{4n} + 1)$.

6. It is stated on p. 154 that Lemmas IV. and V. may be proved by a method similar to that employed for Lemma III. However, this is not the case with Lemma V. In fact, suppose that $\zeta_{56} \neq 0$, so that $\xi_5 = \xi_6 = 0$. Applying a product of the generators C, we may take $\xi_i = 0$ $(i = 1, \ \ldots, \ 4)$, and obtain a transformation S_1, leaving fixed x_1, x_2, x_3, x_4, and replacing x_5 by $\zeta_{56} z_{56}$. By the argument in the text, S_1 replaces x_6 by

$$\sum_{i=1}^{6} \xi_i' x_i + \zeta_{56}' z_{56} \quad (\xi_5' \zeta_{56}' = 0, \ \xi_6' \zeta_{56}' = 0).$$

Now $A_{56,1}$ leaves fixed $x_1, \ \ldots, \ x_4, \ z_{56}$, and replaces x_5 by $x_5 + x_6$. Hence $S_1 A_{56,1}$ replaces x_5 by

$$\sum_{i=1}^{6} \xi_i' x_5 + (\zeta_{56} + \zeta_{56}')\, z_{56}.$$

The corresponding conditions

$$\xi_5' (\zeta_{56} + \zeta_{56}') = 0, \quad \xi_6' (\zeta_{56} + \zeta_{56}') = 0,$$

combined with the earlier ones, give $\xi_5' = \xi_6' = 0$. Then $|S_1| = 0$, Hence must $\zeta_{56} = 0$. Lemma V. should therefore read:

If a transformation of $G(F)$ *leaves fixed* x_1, x_2, x_3, *and* x_4, *it replaces* χ_5 *by* $\sum_{i=1}^{6} \xi_i \chi_i$. *Inversely, there exists in* $G(F)$ *such a transformation for* $\xi_1, \ldots, \xi_4$ *arbitrary,* ξ_5 *and* ξ_6 *not both zero.*

When F is the $GF[p^n]$, the number of functions $\Sigma \xi_i x_i$ is $p^{4n}(p^{2n}-1)$. This factor must therefore replace the factor $p^{4n}(p^n-1)(p^n+2)$, bottom of p. 158. The correct formula* for the order of the group G in the $GF[p^n]$ is

$$p^{36n}(p^{12n}-1)(p^{9n}-1)(p^{8n}-1)(p^{6n}-1)(p^{5n}-1)(p^{2n}-1).$$

The minimum order, given by $p^n=2$, is

$$2^{36}.3^6.5^2.7^3.13.17.31.73.$$

7. The inverse of the general linear transformation S leaving the cubic form (1) invariant may be found by the following direct method. Let C' denote the form (1) written in the transformed variables x', y', z'. Let S be given the notation on p. 167. Since $C=C'$ under S, we have

$$x_l = \frac{\partial^2 C}{\partial y_k \partial z_{lk}} = \frac{\partial^2 C'}{\partial y_k \partial z_{lk}}. \tag{7}$$

In the final term (7) the coefficient of x_i' is

$$\Sigma\left(\frac{\partial y_j'}{\partial y_k}\frac{\partial z_{ij}'}{\partial z_{lk}} + \frac{\partial y_j'}{\partial z_{lk}}\frac{\partial z_{ij}'}{\partial y_k}\right) = \Sigma(\mu_{jk}n_{ijlk} + \nu_{jlk}m_{ijk}), \tag{8}$$

summed for $j=1, \ldots, 6$; $j \neq i$. The coefficient of y_i' is

$$\Sigma\left(\frac{\partial x_j'}{\partial y_k}\frac{\partial z_{ji}'}{\partial z_{lk}} + \frac{\partial x_j'}{\partial z_{lk}}\frac{\partial z_{ji}'}{\partial y_k}\right) = \Sigma(M_{jk}n_{jilk} + N_{jlk}m_{jik}). \tag{9}$$

Hence the coefficients (say L_{li}' and M_{li}') of x_i' and y_i' in x_l are given by the final sums (8) and (9), respectively. Either sum must have the same value for $k=1, \ldots, 6$; $k \neq l$.

The results on p. 169 may therefore be simplified. The partial sums for each value of t are all equal.

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* In all the known systems of simple groups, the order is a product of literal factors p^k, $p^k \pm 1$. The suspicious factor p^n+2 led to the present revision.

The University of Chicago,
August, 1906.

TRANSITIVE GROUPS OF DEGREE $p=2q+1$, p AND q BEING PRIME NUMBERS.

By G. A. Miller.

THE groups of this degree were considered by various authors, especially by Mathieu and by de Seguier.* Since they furnish the simplest examples of constructing all the possible transitive groups of a fairly large degree, it seems especially desirable to exhibit the simplest possible general method of finding all such groups when the degree is given. While the method which we shall give is partly tentative, yet it involves less than half the number of trials used in the latter of the articles mentioned above. Moreover, we shall complete the enumeration of such groups to degree 107 by determining all the possible transitive groups of degree 83. We shall also be able to give some important new facts about the known groups of these degrees. Among these are the following:—The four-fold transitive group of degree 23 is perfect as well as simple, and by means of this result it is easy to prove that the five-fold transitive group of degree 24 is also perfect, while the five-fold transitive group of degree 12 has a group of twice its own order for its group of isomorphisms.

It has been proved by means of the most elementary considerations that the subgroups of order p in any transitive group of degree p generate a simple group, and that each of these subgroups is transformed into itself by at least $2p$ substitutions of the group unless the group is cyclic. Moreover, the metacyclic group of degree p contains just as many transitive subgroups as there are different divisors of $p-1$. In the groups under consideration the metacyclic groups involve four transitive groups of degree $p>5$ (one of each of the orders p, $2p$, qp, and $2qp$) and this is the smallest possible number of such groups of any prime degree greater

* Mathieu, *Liouville*, Vol. XVIII. (1873), p. 25; de Segnier, *ibid.*, Vol. VIII. (1902), p. 280.

than 5. If we add the alternating and the symmetric group we have the theorem that there are at least 6 transitive groups of every prime degree greater than 5, and if there are only 6 transitive groups of such a degree it must be of the form $2q+1$. It was first proved by Jordan that there are only 6 transitive groups of each of the degrees 47 and 59. If a transitive group of any prime degree contains only one regular subgroup it is contained in the metacyclic group, and hence we need to consider only those groups of degree p which involve more than one subgroup of order p but do not involve the alternating group of degree p. In what follows such a group will be represented by G. Its order evidently exceeds $p(p+1)$.

If G involved negative substitutions its positive substitutions would constitute a subgroup of half its order. Hence we shall assume that G is positive unless the contrary is stated. It will be found that G is necessarily positive for all possible values of $p<107$ which are of the given form. As the substitutions of order 2 in the metacyclic group of degree p are negative, it follows that a subgroup of order p is transformed into itself by exactly qp substitutions of G. Moreover, the subgroup (G_1) of degree $2q$ which is composed of all the substitutions of G, omitting a given letter, cannot involve any cycle of order q*, and hence all its subgroups of order q are conjugate as the order of G_1 cannot be divisible of q^2.

We shall now prove that each subgroup of order q in G_1 is transformed into itself by more than its own substitutions. If this were not the case G_1 would transform its subgroups of order q according to a transitive group whose degree (n) would be equal to the number of these subgroups and whose class would be $n-1$. Hence all its substitutions, except those of order q, would constitute a regular invariant subgroup of order n. This is a special case of the theorem: *If any group involves a subgroup of prime order q which is transformed into itself only by its own substitutions all its substitutions whose orders are not divisible by q constitute an invariant subgroup of index q.*† From this theorem it follows directly that *a non-cyclic group of degree* $2q$ *which involves a cyclic subgroup of order q which is transformed into itself only by its own substitutions is transitive and involves an invariant abelian subgroup of order* 2^a *and of type* (1, 1, 1, ...).

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* *Bulletin of the American Mathematical Society*, Vol. IV. (1898), p. 141.
† Cf. Frobenius, *Berliner Sitzungsberichte*, 1901, p. 1220.

It should be observed that the last theorem stated above is not essential for our present purpose but is given for its own sake. If a subgroup of order q and degree $2q$ would be transformed into itself by only its own substitutions under G_1 it would also be transformed into itself by only q substitutions of G. This is impossible as G cannot involve an invariant subgroup of index q, since in such a subgroup the cyclic subgroups of order p would be transformed into themselves by only p substitutions. As each subgroup of order q is transformed into itself by more than q substitutions of G_1, and as G_1 involves only positive substitutions, it follows that these substitutions do not interchange the systems of intransitivity of the subgroup of order q, and that they may be found by establishing a simple isomorphism between a transitive subgroup of the metacyclic group and itself, written in a distinct set of letters. Since the group of isomorphisms of each of these non-cyclic subgroups is of order $q(q-1)$ *this isomorphism can be established in exactly q different ways when both the subgroup of order q and the number of substitutions are given.*

We are now in position to state the general method of finding all the possible transitive groups of degree $p=2q+1$. If any group exists in addition to the six which are well known it may be found as follows:—Select a cyclic substitution (t_1) of degree p arbitrarily, write its fourth power under it and thus find a substitution (t_2) of order q which satisfies the equation $t_2^{-1}t_1t_2=t_1^4$. Then find by inspection the negative substitution which transforms t_1 into t_1^{-1} and involves all the letters which are in t_2. Find the prime divisors of $q-1$ and represent them by $d_1, d_2, \ldots, d_\lambda$. Determine the substitution t_3' which transforms each of the two cycles of t_2 into a power belonging to exponent $d_\alpha(\alpha=1, 2, \ldots, \lambda)$ and omits a transposition of t_0. Since t_0 is commutative with t_2 it is also commutative with t_3'. Finally, find the $\frac{1}{2}(q-1)$ conjugates of t_3' under the first $\frac{1}{2}(q-1)$ powers of one of the cycles of t_2. These conjugates and t_3' are the $\frac{1}{2}(q+1)$ substitutions which are to be used for t_3 in connection with the factor d_α. The group generated by t_1 and t_3 is simple and of composite order. If it is not the alternating group of degree p it is one of the G's in question.

If t_1, t_3 generate the alternating group the fact will generally become evident from the form of the product t_1t_3, since the class of G must exceed $n/4-1$, as it is multiply transitive.*

* Bochert, *Mathematische Annalen*, Vol. XL. (1892), p. 179.

For small values of p other theorems relating to primitive groups may be used to prove that t_1, t_3 generate the alternating group in case they have this property, and in all cases it will appear from a few trial products whenever t_1, t_3 generate the alternating group of degree p. The group generated by t_1, t_3 includes t_2, and when it is not the alternating group it cannot be invariant under a larger group of degree p except when t_3 is commutative with t_0, since such a group cannot involve a cycle of order q. When t_0 and t_3 are commutative G is invariant under one and only one larger group of degree p. The order of this group is twice that of G. When G is contained non-invariantly in a larger group the order of this group must be at least $p + 1$ times that of G, since it could not contain less than $p + 1$ conjugates of G.

To illustrate the simplicity of this method we shall redetermine the possible transitive groups of degrees 5, 7, and 11, giving all the details. When $p = 5$, $q = 2$, and since $q - 1$ has no prime divisor, there is no G of degree 5. That is, there is no transitive group of degree 5 which involves more than one subgronp of order 5 but does not involve all the positive substitutions of this degree. When $p = 7$, $q = 3$ and $q - 1$ has one prime divisor, viz. 2. If we let

$$t_1 = abcdefg, \quad t_2 = bec.dfg, \quad t_0 = bg.cf.de, \quad t_3' = ce.df$$

it is evident that we may select

$$ce.df \text{ and } ce.dg$$

as the $\frac{1}{2}(q + 1) = 2$ substitutions which may be used for t_3.

It is easy to see that t_1 and $ce.df$ generate the alternating group of degree 7. In fact, the group generated by these two substitutions is transformed into itself by t_0, and t_0, t_1, $ce.df$ generate the symmetric group since their continued product is $ab.cge$. Hence we have proved that every transitive group of degree 7, which contains more than one subgroup of order 7, includes the group generated by t_1, $ce.dg$. As this group contains at least 8 subgroups of order 7, and involves t_2, its order cannot be less than 168. Since the abelian group of order 8 and of type (1, 1, 1) has a transitive group of order 168 and of degree 7 for its group of isomorphisms, we have proved that there is one and only one G of order 168 and of degree 7.

If there were another G it would involve this simple group of order 168 non-invariantly, since $ce.dg$ is not commutative

with t_0. Hence its order would be at least 8.168 and the order of its G_1 would be at least 192. As G_1 would be positive it would be the alternating group. That is, there is only one G of degree 7. It may be observed that this determination of the transitive groups of degree 7 does not involve any property of the substitution groups of degree 6 except that the order of the alternating group is 360, and hence cannot contain a subgroup whose order is as large as 192. By employing the substitutions groups of degree 6 it is easy to find all the transitive groups of degree 7 without the use of any trial. In the above method there is only one trial involved, viz. finding the product of $t_0 t_1 t_3$. To appreciate the simplicity of this method it may be compared with the one employed by Burnside on p. 206 of his *Theory of Groups*. The method has much in common with the one employed by de Seguier on p. 282 of the article cited above.

Transitive groups of degrees 11 *and* 23.

Let

$$t_1 = abcdefghijk,\ t_2 = befjd.cikhg,\ t_0 = bk.cj.di.eh.fg,$$

$$t_3' = de.fj.cg.hi.$$

The three substitutions which may be used for t_3 are evidently

$$de.fj.cg.hi,\ de.fj.ci.gk,\ de.fj.ch.ik.$$

From the product $t_1 t_0 t_3' = ak.bfj.che.dig$ it follows that t_1, t_3' generate the alternating group of degree 11 and hence *there is no G of degree* 11 *which is invariant under a larger group of this degree.* This statement follows from the general theory. It may also be seen directly, since such a group would involve two substitutions of the forms $de.fj.ci.gk$, $de.fj.ck.gh$, as it would be invariant under t_0. Hence it would involve a cycle of order q. From the product

$$t_1.de.fj.ch.ik = abhk.cejifg$$

it follows that every G of degree 11 contains t_1, t_2, t_3, where $t_3 = de.fj.ci.gk$, as the class of such a group exceeds 4.

Since the group generated by t_1, t_2, t_3 involves at least 12 subgroups of order 11 its order cannot be less than 660. It is

known that the simple group of order 660 involves subgroups of order 60, and hence it can be represented as a transitive group of degree 11. Hence we have proved that t_1, t_2, t_3 generate the simple group of order 660, and every other group of degree 11, which involves more than one subgroup of order 11, involves this simple group non-invariantly, and has for its order a number which cannot be less than $12.660 = 7920$. Moreover, it is known that the five-fold transitive group of degree 12 contains a subgroup of order 7920 and of degree 11. It is therefore only necessary to prove that there cannot be more than one such group. This follows directly from the fact that its maximal subgroup of degree 10 is of order 720, and hence it is a subgroup of the group of isomorphisms of the alternating group of degree 6*. As this group of isomorphisms contains 40 substitutions which transform one of its subgroups of order 5 into itself the given group of order 720 involves 20 such substitutions. This includes all the possible positive substitutions in ten letters which do not generate substitutions whose class is less than 6. Hence the group of order 7920 is completely determined by its subgroup of order 660 and the 20 substitutions which transform one of its subgroups of order 5 into itself, and *there is one and only one transitive group of degree* 11 *and of order* 7920.

If there were another G of degree 11 it would involve more than 12 subgroups of order 660, and hence it would contain at least 24 such subgroups†. Its maximal subgroup of degree 10 would therefore be at least of order 1440, and hence its class could not exceed 4, since it would be positive. This completes the proof of the known theorem: *there are two and only two transitive groups of degree* 11 *which contain more than one subgroup of order* 11 *but do not contain all the positive substitutions of this degree.* It has recently been proved that this simple group of order 7920 is also perfect, but the simple group of order 660 has a group of twice its order for its group of isomorphisms.‡ This determination of all transitive groups of degree 11, which do not come directly under well known theorems, is given mainly with a view to illustrate the general method, but it is so much briefer than the others which have been published that it would appear to be of interest in itself.

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* *Quarterly Journal of Mathematics*, Vol. XXXI. (1899), p. 228; *Ibid.*, Vol. XXVII. (1894), p. 44.

† *Bulletin of the American Mathematical Society*, Vol. IV. (1898), p. 141.

‡ *Messenger of Mathematics*, Vol. XXXVII. (1907), p. 54.

The writer has used the same method to verify the statement of Jordan and de Seguier that there is only one G of degree 23, and that no G exists when p is either 47 or 59. The work of determining all the transitive groups of such a prime degree $p < 83$ has therefore been done by at least two writers. Although the number of transitive groups of degree n generally increases very rapidly with n, yet, when n is a prime of the given form, these investigations exhibit a decrease rather than an increase. The G of degree 23 is the four-fold transitive group of degree (11) which is a maximal subgroup of the five-fold transitive group of degree (12). Its order must therefore be $48.20.21.22.23$. If it were not a perfect group it would contain an intransitive subgroup of degree 23 and of order $48.20.21.22$. This is impossible, since the alternating group is the only positive group of degree 11 whose order is divisible by 7, and hence the four-fold transitive group of degree 23 is perfect as well as the one of degree 11. That the five-fold transitive group of degree 24 is also perfect follows from the facts that it cannot be invariant under a larger group of this degree, and it cannot involve a subgroup of degree 24 and of order $48.20.21.22.23$.

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Transitive groups of degree 83.

When $p = 83$, $q = 41$ and the prime factors of $q - 1$ are 2 and 5. There are therefore $q + 1 = 42$ substitutions which may be used for t_3. These substitutions may readily be written down after t_1 and t_2 have been selected. They are so long and the theory is so simple that it does not appear desirable to reproduce them. It seems only necessary to state that in all cases t_1 and t_3 were found to generate the alternating group, and hence *there are only six transitive groups of degree* 83. In addition to the alternating and the symmetric group they are composed of the metacyclic group and its transitive subgroups. The orders of these four groups are 83, 166, 3403, and 6806 respectively.

University of Illinois,
September, 1907.

ON THE CONSTRUCTION OF A POINTWISE DISCONTINUOUS FUNCTION ALL OF WHOSE CONTINUITIES ARE INFINITIES AND WHICH HAS A FINITE GENERALISED INTEGRAL.

By W. H. Young, Sc.D., F.R.S.

§ 1. THE function

$$f(x) = \Sigma A_n (x - a_n)^{\frac{1}{3}} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and functions of the same type with different indices, where the a's are a countable set dense everywhere in the interval considered and the coefficients A_n are conveniently chosen, have from time to time been used to illustrate various points.* Pompeju has recently shown that it has a positive differential coefficient everywhere given by the series

$$f'(x) = \Sigma \tfrac{1}{3} \frac{A_n}{(x - a_n)^{\frac{2}{3}}} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

obtained by term-by-term differentiation of (1). The points at which this differential coefficient is infinite ($+\infty$) includes, therefore, the points a_n and are dense everywhere; for the rest they form an inner limiting set of content zero.

It is of interest to note, what has, I believe, not been pointed out, that this latter function $f'(x)$ is a lower semi-continuous function whose lower integral is the original function $f(x)$. It is, in fact, the limit of the monotone increasing sequence of functions, continuous in the extended sense, and each integrable, constituted by the partial summations.†

As the function $f'(x)$ is lower semi-continuous, all its infinities are continuities, and since these are dense everywhere, there are, of course, no others. Thus this is an example of a positive function which is both infinite and continuous

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* G. Cantor, *Math. Ann.*, Vol. XIX. (1882), p. 591. Scheefer, *Acta Math.* Vol. V. (1884). Borel, *Leçons sur la théorie des fonctions* (1898), p. 67. W. H. Young, *Mess. of Math.*, Vol. XXXVII., pp. 53, 54. Pompeju, *Math. Ann.* Vol. LXIII. (1907). p. 326.

† W. H. Young, *Proceedings of the Cambridge Philosophical Society*, 1908.

at each point of an everywhere dense set, and yet has a finite integral in the generalised, or Lebesgue, sense.

Before I was acquainted with the paper of Pompeju quoted, I had already built up a function with these properties, and I give this function here. Though certainly not so elegant as the one above given, it has the practical advantage that its value at any assigned point can be assigned with any required degree of accuracy, whereas in the former function we cannot even say whether a point assigned at random is one at which the series converges or not, much less compute the value of the sum of the series. It should be noticed that the function given below is the differential coefficient of its lower integral at all the points of the inner limiting set dense everywhere at which it is infinite, so that the integral function, like that used by Pompeju, has the property of being monotone and having a vertical inflexional tangent in every interval.

§ 2. We have, first of all, to construct an inner limiting set dense everywhere and of content zero. Such a set arises, theoretically, for example, when, as Borel did, we enclose each rational point p/q in an interval of length $2\epsilon/nq^2$, where n is an integer, and so get a set of intervals D_n, which, for all values of n, have as common internal points such a set as is here contemplated*.

Practically it is more convenient to construct the complementary set, which is, of course, a set of the first category of content equal to that of the whole continuum considered. This we proceed to do.

Let G be a set of the first category in the segment (0, 1) of content unity. This may be constructed as follows:—

Let G_1 be the perfect set nowhere dense whose black intervals are got by dividing the interval (0, 1) into three parts, blackening the middle part, dividing each white segment into 3^2 parts and blackening the middle part, dividing each remaining white segment into 3^3 parts, and so on. The content I_1 of G_1 then lies between

$$(1-\tfrac{1}{2}) \text{ and } (1-\tfrac{1}{3}).\dagger$$

G_2 consists of G_1 and the set got by performing a similar division and blackening of intervals in each of the black

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* Borel, *Leçons sur la théorie des functions* (1898), p 44. Young, *Theory of Sets of Points*, Cambridge University Press (1906), Ex. 9, p. 68.

† Young, *Theory of Sets of Points*, Ex. 1, pp. 78, 79, where the ternary numbers corresponding to the points of the perfect set G_1 are given. Similarly the numbers corresponding to the points of G_n may be characterised with the base $2n+1$.

intervals of G_1, taking 5 instead of 3 as basis of division. The content of the set inserted in a black interval of length l, is then lI_2', where I_2' lies between $(1-\frac{1}{4})$ and $(1-\frac{1}{5})$.

At the next stage we take 7, then 9, and so on, as basis of division, and construct the succeeding perfect sets on the same principle as before. It is then clear that the content I_n of the perfect set G_n is equal to

$$I_{n-1}+(1-I_{n-1})\,I_n'=I_n'+(1-I_n')\,I_{n-1}>I_n',$$

where
$$\left(1-\frac{1}{2n}\right)<I_n'<\left(1-\frac{1}{2n+1}\right).$$

These perfect sets being each dense nowhere, the outer limiting set G is a set of the first category, and is of content

$$\operatorname*{Lt}_{n=\infty} I_n=1.$$

§3. Taking this set G, let f_1, f_2, ... be a monotone descending sequence of upper semi-continuous functions defined as follows:—

f_1, and all subsequent f_n's, is to be 1 at all the points of G_1;

f_2, and all subsequent f_n's, is to be $\frac{1}{2}$ at all the points of G_2-G_1;

f_3, and all subsequent f_n's, is to be $\frac{1}{3}$ at all the points of G_3-G_2;

and so on: in the black intervals of G_n, f_n is to be $1/n+1$ for each value of n.

The limiting function, which is of course upper semi-continuous, is then $1/n$ at each point of G_n-G_{n-1}, and zero at all the points of the set complementary to the set G of the first category.

This complementary set is, of course, of the second category, and, therefore, also everywhere dense in the segment (0, 1).

Denoting by F the function which has at every point of the segment (0, 1) the value inverse to that of the limiting function, we see that F has the following properties. It is infinite and continuous in the extended sense at an everywhere dense set of points, whose content is zero, and is discontinuous and finite at every other point of the segment. It is, moreover, a lower semi-continuous function, and is the limiting function of the finite lower semi-continuous functions $F_n=1/f_n$. Both the lower and the upper integral of this function are

infinite, but it is not difficult so to modify our construction as to get a function of the same type whose lower integral is finite.

§ 4. Instead of taking the integers 3, 5, 7, ... $2n+1$, ... as bases of the successive divisions in forming the sets $G_1, G_2, \ldots$, let us take

$$9, 25, 49, \ldots (2n+1)^2, \ldots .$$

We then have

$$I_{n+1} - I_n = (1 - I_n) I'_{n+1} < 1 - I'_n,$$

where

$$I'_n = \left(1 - \frac{1}{(2n+1)^2}\right)\left(1 - \frac{1}{(2n+1)^4}\right)\left(1 - \frac{1}{(2n+1)^6}\right)\ldots .$$

Here

$$I'_n > 1 - \left[\frac{1}{(2n+1)^2} + \frac{1}{(2n+1)^4} + \ldots\right] > 1 - \frac{1}{(2n+1)^2 - 1} > 1 - \frac{1}{4n^2},$$

so that

$$I_{n+1} - I_n < \frac{1}{4n^2}.$$

In the construction of the successive functions f_n we use now instead of n the value $n^{\frac{1}{2}}$.

In calculating the lower integral of $F = 1/f$, so constructed, we use the theorem that the lower integral of the limit of monotone ascending sequence of bounded lower semi-continuous functions is the limit of their lower integrals. Now it is easily seen that all these lower integrals, which are, of course, positive and finite, form a converging sequence of the same type as the successive sums $s_1, s_2, \ldots$ of the series whose general term is $n^{-\frac{3}{2}}$.

In fact, in the case of bounded lower semi-continuous function, the lower integral coincides with the generalised integral.* Hence

$$\int_0^1 F_n(x)\,dx = I_1 . 1 + (I_2 - I_1) 2^{\frac{1}{2}} + (I_3 - I_2) 3^{\frac{1}{2}} + \ldots$$

$$\ldots + (I_n - I_{n-1}) n^{\frac{1}{2}} + (1 - I_n)(n+1)^{\frac{1}{2}}$$

$$< 1 + \frac{1}{4} . 2^{\frac{1}{2}} + \frac{1}{4 . 2^2} . 3^{\frac{1}{2}} + \ldots \frac{1}{4(n-1)^2} n^{\frac{1}{2}} + \frac{1}{4n^2}(n+1)^{\frac{1}{2}}.$$

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* "General Theory of Integration," *Phil. Trans.*, A. 204, 1905, p. 241.

Proceeding to the limit we find that the lower integral $\int_0^1 F(x)\,dx$ is less than the sum of a convergent series, which proves that the lower integral is finite. It is obvious that the upper integral is infinite; in fact, the associated upper limiting function of F is infinite everywhere.

§ 5. To appreciate the bearing of this example on the theory, it should be borne in mind that the function constructed is lower semi-continuous, and that, therefore, its lower integral coincides with its generalised integral. Thus we have proved the existence of a class of pointwise discontinuous functions infinite in every interval and possessing none the less a finite generalised integral. It may be added that the example constructed is typical of all such pointwise discontinuous functions. In fact, the generalised integral of a positive function has necessarily the value $+\infty$ if the points at which it is infinite form a set of positive content; it is therefore *a priori* necessary for a function to belong to the class in question that the points at which it is infinite form a set of content zero. This condition, though necessary, is, of course, not sufficient, as we have seen.

It is scarcely necessary to point out that there is no difficulty in forming totally discontinuous functions unbounded in every interval with a finite lower integral, *e.g.* take the function which is ∞ at all the irrational points and 0 at all the rational points; its associated lower limiting function is 0 everywhere, and its lower integral is also zero: its generalised integral is, however, infinite.

It is remarkable that a pointwise discontinuous function, infinite in every interval, and therefore continuous only at points where it is infinite, should have a finite generalised integral. The infinities which are continuities, reproduce themselves as infinities of the associated lower limiting function; this is therefore just such a lower semi-continuous function as we have constructed.

Göttingen.

THE DEFINITION OF AN INFINITE INTEGRAL AS THE LIMIT OF A FINITE OR INFINITE SERIES.

By T. J. I'a Bromwich and G. H. Hardy.

1. A FINITE integral is an integral of a limited function over a finite range, and is defined in the ordinary manner as the limit of a finite sum

$$\sum_{r=1}^{n} f_r \delta_r.$$

All other integrals are included in the class of *infinite integrals*. Among the infinite integrals which occur in the ordinary applications of the integral calculus we may distinguish two fundamental types.

In an *infinite integral of the first type* the range of integration is finite: we may suppose it to be (0, 1). And the subject of integration is continuous except at one end (say 0) of this interval, while as x tends to 0, the subject of integration tends to ∞ or $-\infty$, or oscillates infinitely: as in

$$\int_0^1 \frac{dx}{\sqrt{x}}, \quad \int_0^1 \log x \, dx, \quad \int_0^1 \frac{1}{x} \sin\left(\frac{1}{x}\right) dx.$$

In an *infinite integral of the second type* the range is infinite, and the subject of integration is continuous throughout any finite interval included in the range. Thus we may take as examples

$$\int_0^\infty e^{-x} dx, \quad \int_0^\infty \frac{dx}{1+x^2}, \quad \int_0^\infty \sin x^2 \, dx.$$

2. An integral of either of these types is defined as a *repeated limit*. Thus the typical integrals

$$\int_0^1 f(x)\, dx, \quad \int_0^\infty f(x)\, dx$$

are defined as

$$\lim_{\varepsilon \to 0} \int_\varepsilon^1 f(x)\, dx, \quad \lim_{X \to \infty} \int_0^X f(x)\, dx,$$

that is, as the limits of a simple limit.

It is common, of course, to meet with integrals which do not strictly belong to either of these types. But as a rule such integrals may be reduced to the sum of a finite number of integrals of one or other type. Thus

$$\int_0^\infty \frac{x^{a-1}dx}{1+x} \quad (0<a<1)$$

may be expressed in the form

$$\int_0^1 \frac{x^{a-1}dx}{1+x} + \int_0^\infty \frac{x^{a-1}dx}{1+x},$$

that is, as the sum of an integral of the first and an integral of the second type.

Occasionally we meet with an integral of a genuinely more complicated type, such as

$$\int_0^\infty \frac{\log\cos^2 x}{1+x^2}\,dx.$$

This is defined as a *triple (repeated) limit.* For it is

$$\lim_{X\to\infty} \int_0^X \frac{\log\cos^2 x}{1+x^2}\,dx,$$

and the integral from 0 to X is itself the sum of a finite number (increasing indefinitely with X) of infinite integrals of the first type.

But integrals even of this degree of complexity are of comparatively rare occurrence in ordinary work, and in what follows we shall confine ourselves exclusively to integrals of the two standard types.

The first type.

3. We shall now consider the standard integrals of the first type,

$$\int_0^1 f(x)\,dx, \tag{1}$$

and the first question which we shall consider is *whether it is possible, under reasonably general conditions, to replace the ordinary definition by means of a double (repeated) limit by a definition by means of a single limit.*

Let us consider first the special integral

$$\int_0^1 \log x\,dx. \tag{2}$$

If $f(x)$ were limited, the simplest definition of (1) would be as

$$\lim (1/n) \sum_1^n f(x_r), \tag{3}$$

where x_r is any point in the interval $[(r-1)/n, r/n]$. It is evident that this definition would fail in the case of the integral (2), since, for any given value of n, x_1 could be chosen so that $(1/n)f(x_1)$ was greater than any assigned number.

It is, however, natural to suppose that the definition may apply if we adopt some restrictive hypothesis as to where, in the interval $[(r-1)/n, r/n]$, x_r is to be chosen: and the most obvious choice is $x_r = r/n$. Our definition would then be as

$$\lim (1/n) \sum_1^n f(r/n). \tag{4}$$

It is easy to verify that, in the case of $f(x) = \log x$, this expression does converge to a limit equal to the value of the integral as ordinarily defined. For

$$\lim (1/n) \sum_1^n \log (r/n) = \lim (1/n) (\log n! - n \log n) = -1,$$

and it is easy to see, by integration by parts, that

$$\int_0^1 \log x dx = -1.$$

4. The most ordinary case is that in which $f(x)$ *is positive and tends steadily to* ∞ *as* x *tends to zero.* It is easy to prove that if this condition is satisfied,

$$\lim_{n \to \infty} (1/n) \sum_1^n f(r/n) = \int_0^1 f(x)\, dx$$

*whenever the latter integral is convergent**, while in the contrary case

$$(1/n) \sum_1^n f(r/n) \to \infty .$$

For it is evident that

$$\int_{1/n}^1 f(x)\, dx \tag{5}$$

* This result is included in a theorem given by de la Vallée-Poussin, *Cours d'Analyse*, t. II. (1906), p. 79.

lies between the two sums

$$(6)\quad \begin{cases} (1/n)\,[f(1/n)+f(2/n)+\ldots+f\{(n-1)/n\}], \\ (1/n)\,[f(2/n)+\ldots+f\{(n-1)/n\}+f(1)], \end{cases}$$

whose difference is

$$(1/n)\,\{f(1/n)-f(1)\}.$$

But it is a well known result (due to Pringsheim) that if $f(x)$ steadily increases as x decreases, and $\int_0^1 f(x)\,dx$ is convergent, then $\lim_{x\to 0} xf(x)=0$. Hence

$$\lim (1/n)\,\{f(1/n)-f(1)\}=0.$$

Hence the two sums (6) tend to the same limit as does the integral (5): and from this our conclusion follows.

On the other hand,

$$(1/n)\,\{f(1/n)+f(2/n)+\ldots+f(1)\}>\int_{1/n}^{1} f(x)\,dx.$$

and so, if $\int_0^1 f(x)\,dx$ is divergent,

$$(1/n)\sum_1^n f(r/n)\to\infty.$$

We may notice that, since

$$\int_{1/m}^{1} f(x)\,dx=\lim_{n\to\infty}(1/n)\sum_1^n f[(1/m)+(r/n)\,\{1-(1/m)\}],$$

the original definition of the integral yields

$$\int_0^1 f(x)\,dx=\lim_{m\to\infty}\lim_{n\to\infty}(1/n)\sum_1^n f[(1/m)+(r/n)\,\{1-(1/m)\}].$$

If we invert the order of these two limits, we obtain

$$\lim_{n\to\infty}(1/n)\sum_1^n f(r/n),$$

which is the limit considered above. Thus the result which we have proved is really one concerning the inversion of two limiting processes.

5. *Examples.* (i) If $s>-1$, we have

$$\int_0^1 x^s dx=\frac{1}{s+1}.$$

Hence, if $s > -1$,

$$\lim (1/n) \sum_1^n (r/n)^s = 1/(s+1)$$

or
$$\lim (1^s + 2^s + \ldots + n^s)/n^{s+1} = 1/(s+1).$$

This proof is of course that ordinarily given when $s > 0$: but the result of § 4 is required to justify it when $0 < s < 1$.

(ii) The values of a number of integrals of the first type are readily calculated by means of the result of § 4. Thus

$$\int_0^1 \log \sin \tfrac{1}{2}\pi x\, dx$$

$$= \lim (1/n) \log \{\sin(\pi/2n) \sin(2\pi/2n) \ldots \sin(n\pi/2n)\}$$

$$= -\log 2.$$

This method of evaluating the integral is given in some text-books, but without any reference to the theoretical difficulty involved.

(iii) Or, again, asymptotic formulæ may be deduced (as in Ex. i.) from known values of integrals. Thus the assumptions

$$f(x) = x \log x, \quad (\log x)^2$$

lead to the formulæ

$$1^1 2^2 \ldots n^n = e^{\frac{1}{2}n^2 \log n - \frac{1}{4}n^2 (1+\epsilon_n)},$$

$$\sum_1^n (\log r)^2 = n (\log n)^2 - 2n \log n + 2n (1 + \epsilon_n);$$

where ϵ_n denotes a function of n, which tends to zero when n tends to infinity; it is, of course, not the same function in the two cases.

(iv) It is perhaps worth while, before passing on, to give a simple example to show the limitations of the method here considered.

Suppose that for $x = 2^{-\nu}$ ($\nu = 0, 1, 2, \ldots, \infty$), $f(x) = 2^\nu$. Then the sum

$$(1/2^n) \sum_1^{2^n} f(r/2^n)$$

includes the term $(1/2^n)f(1/2^n)=1$, and others, and so is greater than unity. Hence

$$(1/m)\sum_1^m f(r/m)$$

certainly cannot tend, as $m\to\infty$, to a limit less than 1.

But it is easy to define a function $f(x)$ (graphically or otherwise), which is positive and continuous for $x>0$, has the special values assigned above, and yet is such that $\int_0^1 f(x)\,dx<\epsilon$, where ϵ is any assigned positive number.

6. It might be thought that, if we replace the ordinary definition of the integral (1) as a repeated limit by the definition of a simple limit which § 4 shows to be equivalent, we could simplify the treatment of certain cases of the ordinary double limit problems of the integral calculus, such as inversion of two integrations or integration of an infinite series. A typical case is provided when we try to justify the equation

$$\int_0^1 \frac{dx}{\sqrt{x}} = \Sigma \frac{1.3\ldots 2n-1}{2.4\ldots 2n}\int_0^1 (1-x)^n\,dx$$

$$= \Sigma \frac{1}{n}.\frac{1.3\ldots 2n-1}{2.4\ldots 2n},$$

From the abstract theoretical point of view such a problem certainly is simplified. For we have now only to deal with the permutation of *two* limit processes instead of three—the expression

$$\int_0^1 \Sigma \frac{1.3\ldots 2n-1}{2.4\ldots 2n}x^n dx$$

being, when the ordinary definitions are adopted, a *triple* limit.

But the criteria already established for dealing with such cases* are extremely simple, and to use the simpler definition of the integral does not seem to be of any material advantage.

7. *Definition of the integral of the first type as the limit of an infinite series.* It is also possible to use the definition

$$\int_0^1 f(x)\,dx = \lim \Sigma_0^\infty (x_{\nu-1}-x_\nu)f(x_{\nu-1})$$

* See, *e.g.* Hardy, *Messenger of Mathematics*, Vol. XXXV., p. 126; Bromwich *ibid.*, Vol. XXXVI., p. 1, and *Infinite Series*, pp. 448 *et seq.*

where $x_0 (= 1), x_1, x_2, \ldots$ is a descending sequence whose limit is zero, the terms of the sequence depending on a parameter g in such a way that

$$\lim_{g \to 1} (x_{\nu-1} - x_\nu) = 0$$

for all values of ν. The only case of interest is that in which $x_\nu = g^\nu$, where $0 < g < 1$, and it is easy to prove, by means of considerations similar to those of § 4, that when $f(x)$ is positive and tends steadily to ∞ as $x \to 0$

$$(7) \quad \lim_{g \to 1} (1-g) \{f(1) + gf(g) + g^2 f(g^2) + \ldots\} = \int_0^1 f(x)\, dx$$

whenever the integral is convergent.

It is instructive to evaluate a few simple integrals, such as

$$\int_0^1 x^{-s} dx, \quad \int_0^1 \log x dx,$$

by means of this formula.

The second type.

8. We consider next the standard integral of the second type, viz.

$$(8) \qquad \int_0^\infty f(x)\, dx.$$

Let us suppose that $f(x)$ is positive and tends steadily to zero as x tends to infinity. Then it is easy to frame definitions of the integral as a simple limit. For example, we may define it as

$$\lim_{h \to 0} h \{f(0) + f(h) + f(2h) + \ldots + f(nh)\},$$

where n is such a function of h that $nh \to \infty$ as $h \to 0$. But such definitions do not, it this case, appear to possess any special points of interest.

9. The more interesting question, in this case, is that of the definition of the integral (7) as the limit of an *infinite* series. This question has been discussed by Dini* and by

* *Grundlagen*, pp. 453 *et seq.*

Hobson*, but from too general a point of view to be of much practical interest.

The principal results there proved are the following; in them δ_ν denotes an interval depending upon a parameter α, and, for any value of ν, tending to zero with α, and U_ν, L_ν denote the upper and lower limits $f(x)$ in δ_ν:—

(1) *If a system* (δ_ν) *can be found such that* $\Sigma U_\nu\delta_\nu$, $\Sigma L_\nu\delta_\nu$ *are convergent for* $\alpha>0$, *and tend, as* $\alpha\to 0$, *to the same limit* A, *then* $\int_0^\infty f(x)\,dx$ *is convergent and has the value* A.

(2) *If* $\int_0^\infty f(x)\,dx$ *is convergent and has the value* A *it is possible to find a system* (δ_ν) *having the properties described in* (1) *above.*

10. The most interesting set of intervals (δ_ν) is obviously that divided by the points

$$0,\ h,\ 2h,\ 3h,\ \ldots,$$

where h is a parameter which tends to zero. It is, however, easy to see that the fact that $f(x)$ is positive and continuous, and $\int_0^\infty f(x)\,dx$ convergent, is not enough to ensure the convergence of

$$h\Sigma f(nh),$$

for all positive values of h.

For let $f(x)=1$ whenever $x=2^\nu$. Then if $h=2^{-w}$, the series includes the terms

$$f(1),\ f(2),\ f(4),\ \ldots,$$

and is therefore divergent. But it is easy to define a continuous function which has the values prescribed above, and yet is such that $\int_0^\infty f(x)\,dx$ is convergent and less than any assigned positive number ϵ.

Let us, however, assume that $f(x)$ *steadily decreases* as x increases. Going back for a moment to the general case, let $x_0(=0)$, x_1, x_2, ... be the points of division of the intervals. Then it is obvious that

$$(x_{n+1}-x_n)f(x_n)\geqq\int_{x_n}^{x_{n+1}}f(x)\,dx\geqq(x_{n+1}-x_n)f(x_{n+1}),$$

* *Theory of Functions of a real Variable*, pp. 399 *et seq.*

and therefore

$$\Sigma_0^\infty (x_{n+1}-x_n) f(x_n) \geqq \int_0^\infty f(x)\,dx \geqq \Sigma_0^\infty (x_{n+1}-x_n) f(x_{n+1})$$

in so far as these inequalities have a meaning: that is to say, the convergence of the integral ensures that of the second series and the second part of the inequality, and the convergence of the first series ensures both inequalities.

It is easy to see, by an example, that the convergence of the integral does not ensure that of the first series. Take, *e.g.* $f(x)=1/(1+x)^2$ and $x_n=2^{2^n}h$. When h is fixed, we have, for sufficiently large values of n,

$$(x_{n+1}-x_n) f(x_n) > Kh2^{2^n}(2^{2^n}-1)/(h2^{2^n})^2 > K/h,$$

where K is a constant as nearly equal to unity as we please.

However, in the case in which $x_n=nh$, the monotonic character $f(x)$ and the convergence of the integral, or of either series, is sufficient to ensure that all three converge and that the limit of either series as $h\to 0$ is equal to the value of the integral.

For if $h\{f(h)+f(2h)+\ldots\}$ is convergent, so also is

$$h\{f(0)+f(h)+f(2h)+\ldots\},$$

and the difference of their sums, viz. $hf(0)$, tends to zero as $h\to 0$. Thus

$$\lim_{h\to 0} h\Sigma_0^\infty f(nh) = \int_0^\infty f(x)\,dx. \tag{9}$$

11. If the lower limit of the integral is a positive number c, and $g>1$, it is easy to prove in the same sort of way that

$$\lim_{g\to 1} c(g-1)\{f(c)+gf(cg)+g^2f(cg^2)+\ldots\} = \int_c^\infty f(x)\,dx$$

whenever $f(x)$ is monotonic and the integral convergent.

12. *Examples.* (i) The formulæ

$$\int_0^\infty xe^{-x}dx = 1, \quad \int_c^\infty x^{-s}dx = c^{-(s-1)}/(s-1) \quad (s>1)$$

are easily verified by using the results of § 10 and § 11 respectively.

(ii) The application of § 10 to

$$\int_0^\infty \frac{dx}{(a+x)^s} \quad (s>1)$$

leads to the formula

$$\lim_{h\to 0} h\left\{\frac{1}{a^s}+\frac{1}{(a+h)^s}+\frac{1}{(a+2h)^s}+\ldots\right\}=\frac{1}{(s-1)\,a^{s-1}}.$$

(iii) Its application to the integral $\int_0^\infty x^{k-1}e^{-x}dx$ leads to the formula

$$\lim h^k(e^{-k}+2^{k-1}e^{-2k}+3^{k-1}e^{-3k}+\ldots)=\Gamma(k),$$

or
$$\lim_{x\to 1}(1-x)^k(x+2^{k-1}x^2+3^{k-1}x^3+\ldots)=\Gamma(k).$$

(iv) It is easy to verify that the equation (9) sometimes holds when $f(x)$ is not monotonic or even essentially positive. Consider, for example, the integrals

$$\int_0^\infty \frac{\sin x}{x}\,dx,\quad \int_0^\infty\left(\frac{\sin x}{x}\right)^2 dx,$$

each of which is equal to $\frac{1}{2}\pi$. The formula (9) leads to the results

$$\lim h\left(1+\frac{\sin h}{h}+\frac{\sin 2h}{2h}+\ldots\right)=\tfrac{1}{2}\pi,$$

or
$$\lim(\sin h+\tfrac{1}{2}\sin 2h+\ldots)=\tfrac{1}{2}\pi,$$

and
$$\lim\frac{1}{h}\left(\sin^2 h+\frac{\sin^2 2h}{2^2}+\ldots\right)=\tfrac{1}{2}\pi,$$

which are easily verified.

13. Such cases as those of the last example fall quite outside the scope of the results proved by Dini and Hobson or in the previous paragraphs of this paper, and require theorems of a much more special character. We shall prove the following theorem:—

If $\phi(x)$ is a continuous function of x which tends steadily to zero as $x\to\infty$, then

$$\lim_{h\to 0} h\Sigma_0^\infty\phi(nh)\sin nh=\int_0^\infty\phi(x)\sin x\,dx.$$

This theorem is a special case of the following more general theorem: *if* (1) *$\phi(x)$ is a continuous function of x which tends*

steadily to zero as $x \to \infty$, (2) $F(x)$ *is a continuous function of* x, *and*

$$h \left| \sum_{n_1}^{n_2} F(nh) \right| < K$$

for all positive integral values of n_1 *and* n_2 $(n_2 > n_1)$ *and for all sufficiently small positive values of* h, *say for* $0 < h \leqq \alpha$, *then*

$$\lim_{h \to 0} h \Sigma_0^\infty \phi(nh) F(nh) = \int_0^\infty \phi(x) F(x)\, dx.$$

For, under the conditions stated, the series on the left is certainly convergent. Moreover, if ξ is any fixed number

$$\int_0^\xi F(x)\, dx = \lim_{h \to 0} h \sum_0^\nu F(nh),$$

where ν is the greatest integer such that $\nu h < \xi$. It follows that

$$\left| \int_0^\xi F(x)\, dx \right| \leqq K,$$

and so

$$\left| \int_\xi^{\xi'} F(x)\, dx \right| \leqq 2K$$

for all values of ξ and ξ' such that $0 < \xi < \xi'$. Hence the integral $\int_0^\infty \phi(x) F'(x)\, dx$ is convergent.

Moreover*

$$h \left| \sum_{n_1}^{n_2} \phi(nh) F(nh) \right| \leqq K\phi(\xi),$$

if $n_1 h \geqq \xi$, nd

$$\left| \int_\xi^{\xi'} \phi(x) F(x)\, dx \right| \leqq 2K\phi(\xi).$$

In these inequalities we may replace n_2 and ξ by ∞, and if we choose ξ so that $\phi(\xi) < \epsilon$, we have, writing $f(x)$ for $\phi(x) F(x)$,

$$(10) \quad h \left| \sum_{n_1}^{\infty} f(nh) \right| < K\epsilon, \quad \left| \int_\xi^\infty f(x)\, dx \right| < 2K\epsilon,$$

provided $n_1 h \geqq \xi$.

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* Bromwich, *Infinite Series*, pp. 50 and 426.

Suppose that $(n_1-1)\,h<\xi\leqq n_1 h$. Then

$$(11)\qquad \left|h\sum_0^{n_1-1} f(nh)-\int_0^{\xi} f(x)\,dx\right|$$

$$\leqq\left|h\sum_0^{n_1-1} f(nh)-\int_0^{(n_1-1)h} f(x)\,dx\right|+\left|\int_{(n_1-1)h}^{\xi} f(x)\,dx\right|$$

$$<\omega\xi+2K\phi\,(\xi-h),$$

where ω is the greatest oscillation of $f(x)$ in any one of the n_1-1 intervals

$$(0,h);\ (h,2h);\ \ldots;\ \{(n_1-2)\,h,(n_1-1)\,h\}.$$

But, ξ having been fixed, we can choose δ so that

$$(12)\qquad \omega<\epsilon/\xi,\quad \phi\,(\xi-h)<\phi\,(\xi)+\epsilon<2\epsilon$$

for $0<h\leqq\delta$. And then, from (10), (11), and (12), we obtain

$$|\,h\Sigma_0^{\infty}f(nh)-\textstyle\int_0^{\infty}f(x)\,dx\,|<(7K+1)\,\epsilon$$

for $0<h\leqq\delta$, which proves the theorem.

The simplest case of this theorem is obtained by supposing

$$F(x)=\sin x.$$

Then

$$h\sum_{n_1}^{n_2}F(nh)=(\tfrac{1}{2}h/\sin\tfrac{1}{2}h)\,\{\cos(n_1-\tfrac{1}{2})\,h-\cos(n_2+\tfrac{1}{2})\,h\}$$

and $\frac{1}{2}h\operatorname{cosec}\frac{1}{2}h<\frac{6}{5}$ if $0<h<2$*, so that we may take $K=1\frac{2}{5}$. Similarly, of course, the theorem holds if $F(x)=\cos x$.

14. When we try to apply the result of the last section to such an integral as

$$\int_0^{\infty}\frac{\sin x}{x}\,dx,$$

we are met by the difficulty that $\phi\,(x)=1/x\to\infty$ as $x\to 0$.

---

* For $(\sin x)/x>(\sin 1)/1>\frac{6}{5}$ if $0<x<1$.

This difficulty, however, is slight and not essential.

Let us suppose that the conditions of § 13 are satisfied except that $\phi(x) \to \infty$ as $x \to 0$, in such a way that $f(x)$ is monotonic near $x = 0$ and $\int_0 f(x)$ convergent.

Let α be any positive number such that $f(x)$ is monotonic in $(0, \alpha)$, and let λ be the greatest integer such that $(\lambda-1)h < \alpha$. Also let $(\lambda - 1)\, h = \alpha_1$. By § 4, we can choose δ so that

$$\left| h \sum_1^{\lambda-1} f(nh) - \int_0^{\alpha_1} f(x)\, dx \right| < \epsilon$$

for $0 < h \leqq \delta$. Also

$$\left| \int_{\alpha_1}^{\alpha} f(x)\, dx \right| < (\alpha - \alpha_1)\, \phi(\alpha - h) < Kh.$$

From these two inequalities it follows that

$$\lim_{h \to 0} h \sum_1^{\lambda-1} f(nh) = \int_0^{\alpha} f(x)\, dx.$$

We can now show, by an argument in all essentials the same as that of § 13, that

$$\lim_{h \to 0} h \sum_{\lambda}^{\infty} f(nh) = \int_{\alpha}^{\infty} f(x)\, dx.$$

Hence $$\lim h \Sigma_1^{\infty} f(nh) = \int_0^{\infty} f(x)\, dx.$$

15. The most interesting examples of §§ 13, 14 are obtained from the integrals

$$\int_0^{\infty} (\sin x / x^s)\, dx = \Gamma(1-s) \cos \tfrac{1}{2} s\pi,$$

$$\int_0^{\infty} (\cos x / x^s)\, dx = \Gamma(1-s) \sin \tfrac{1}{2} s\pi,$$

where $0 < s < 1$. These lead to the results

$$\lim h^{1-s} \{(\sin h / 1^s) + (\sin 2h / 2^s) + \ldots\} = \Gamma(1-s) \cos \tfrac{1}{2} s\pi,$$

$$\lim h^{1-s} \{(\cos h / 1^s) + (\cos 2h / 2^s) + \ldots\} = \Gamma(1-s) \sin \tfrac{1}{2} s\pi.$$

The first of these equations holds also when $s = 1$, if we replace

$$\Gamma(1-s) \cos \tfrac{1}{2} s\pi$$

by $$\lim_{s \to 1} \Gamma(1-s) \cos \tfrac{1}{2} s\pi = \tfrac{1}{2}\pi.$$

These results are, of course, closely connected with the known theorems as to the behaviour of the function

$$x + (x^2/2^2) + (x^3/3^3) + \ldots$$

when $x \to 1$ along a path lying inside the circle of convergence: in fact, they define its behaviour when $x \to 1$ *along* the circle.

16. The integral $\int_0^\infty (\sin x/x)^2 dx$, considered in Ex. 4, § 12, falls outside the scope of § 13, the subject of integration, though not monotonic, being positive, and the convergence of the integral being, of course, absolute.

Such cases as this may be dealt with by the following theorem:—*if* $\phi(x)$ *is positive and continuous, and tends steadily to zero as* $x \to \infty$, *in such a way that*

$$\int_0^\infty \phi(x)\, dx$$

is convergent, and $F(x)$ *is a continuous function whose modulus never exceeds a constant* K, *then*

$$\lim_{h \to 0} \Sigma_0^\infty \phi(nh) F(nh) = \int_0^\infty \phi(x) F(x)\, dx.$$

Let $f(x) = \phi(x) F(x)$, as before. The series and integral

$$\Sigma^\infty f(nh), \quad \int^\infty f(x)\, dx$$

are obviously both absolutely convergent. Moreover

$$\begin{aligned} |h \overset{\infty}{\underset{n_1}{\Sigma}} f(nh)| &< Kh \overset{\infty}{\underset{n_1}{\Sigma}} \phi(nh) \\ &< Kh \left\{ \phi(n_1 h) + \int_{n_1}^\infty \phi(th)\, dt \right\} \\ &< K \left\{ h\phi(n_1 h) + \int_{n_1 h}^\infty \phi(u)\, du \right\}. \end{aligned}$$

Choose ξ so that

$$\phi(\xi) < \epsilon, \quad \int_\xi^\infty \phi(u)\, du < \epsilon.$$

Then, if $h<1$ and $n_1 h \geqq \xi$, we have clearly

$$|h\sum_{n_1}^{\infty} f(nh)| < 2K\epsilon,$$

$$\left|\int_{\xi}^{\infty} f(x)\,dx\right| < K\epsilon.$$

These inequalities are precisely similar to the inequalities (10) of § 13, and the proof of the theorem may now be completed in the same way.

Moreover the theorem may be extended (as that of § 13 was extended in § 14) so as to cover the case in which $\phi(x) \to \infty$ as $x \to 0$, in such a way that $f(x)$ is monotonic near $x=0$ and $\int_0 f(x)\,dx$ convergent.

§ 17. The results of the preceding section enable us to deal with the case in which

$$f(x) = \phi(x)\,\Theta(\sin^2 x),$$

where Θ is any continuous function. When Θ is a *polynomial*, such as $\sin^2 x$, the result may also be deduced from §§ 13, 14 by expressing Θ as a sum of cosines of multiples of x—for example, $\sin^2 x$ as $\frac{1}{2}(1-\cos 2x)$.

The results of §§ 13, 14 may be similarly extended to the case in which

$$f(x) = \phi(x)\,(\sin x \text{ or } \cos x)\,\Theta(\sin^2 x),$$

where Θ is a polynomial. When Θ is not a polynomial the matter is not so simple (see § 19).

§ 18. It is also interesting to observe that if we assume that $\phi'(x)$ as well as $\phi(x)$ is monotonic and continuous, we can deduce the result of § 13

$$h\Sigma_0^{\infty}\phi(nh)\sin nh \to \int_0^{\infty}\phi(x)\sin x\,dx$$

from an obvious extension of the result of § 16. For, let

$$\Delta\phi(nh) = \phi(nh) - \phi\{(n+1)h\} = h\phi'(nh+\theta),$$

where $0 < \theta < h$. Then

$$h \Sigma_0^\infty \sin nh\, \phi(nh) = h \Sigma_0^\infty (\sin h + \sin 2h + \ldots + \sin nh)\, \Delta\phi(nh)$$
$$= \tfrac{1}{2} h \cot \tfrac{1}{2} h \{\phi(0) - \Sigma_0^\infty h \cos nh\, \phi'(nh + \theta)\}$$
$$+ \tfrac{1}{2} h \Sigma_0^\infty h \sin nh\, \phi'(nh + \theta).$$

Now $\int^\infty \phi'(x)\, dx$ is convergent, and the limit of the last expression is easily seen to be

$$\phi(0) - \int_0^\infty \phi'(x) \cos x\, dx = \int_0^\infty \phi(x) \sin x\, dx.$$

§ 19. As was remarked at the end of § 17, we find ourselves faced by more serious difficulties if we try to prove that

$$\lim_{h \to 0} h \Sigma_0^\infty \phi(nh) \sin nh\, \Theta(\sin^2 nh) = \int_0^\infty \phi(x) \sin x\, \Theta(\sin^2 x)\, dx,$$

$\phi(x)$ being subject only to the conditions of §§ 13, 14 and not to the more stringent conditions of § 16, while Θ is any continuous function and not merely a polynomial.

In fact it is open to question whether the series on the left is convergent for all positive values of h (though the integral is certainly convergent).

We can, however, show that it is possible to choose a particular sequence of values of h, tending to zero, such that the series converges and has a limit which is equal to the integral.

Let $$F(x) = \sin x\, \Theta(\sin^2 x).$$

We shall prove that if $F(x)$ is continuous (a more general condition than that of the continuity of Θ), then

$$\lim_{q \to \infty} (p\pi/q) \Sigma \phi(np\pi/q)\, F(np\pi/q) = \int_0^\infty \phi(x)\, F(x)\, dx,$$

where p is a fixed odd integer and q tends to ∞ through integral values.

For

$$(p\pi/q) \sum_{n=0}^{\infty} \phi(np\pi/q)\, F(np\pi/q)$$
$$= (p\pi/q) \sum_{\mu=0}^{q-1} \sum_{r=0}^{\infty} F\{(p\pi/q)(rq + \mu)\}\, \phi\{(p\pi/q)(rq + \mu)\}$$
$$= (p\pi/q) \sum_{\mu=0}^{q-1} F(p\pi\mu/q) \sum_{r=0}^{\infty} (-1)^r \phi\{(p\pi/q)(rq + \mu)\},$$

the convergence of the original series following from that of each of the q infinite series in the last expression.

Let

$$\psi(x) = \sum_{r=0}^{\infty} (-1)^r \phi\{p\pi(r+x)\}.$$

Since ϕ is monotonic this series is uniformly convergent, and $\psi(x)$ continuous, throughout the interval $0 \leqq x \leqq 1$.*

Also

$$\lim_{q\to\infty} (p\pi/q) \sum_{\mu=0}^{q-1} F(p\pi\mu/q)\,\psi(\mu/q)$$

$$= p\pi \int_0^1 F(p\pi\xi)\,\psi(\xi)\,d\xi$$

$$= p\pi \int_0^1 F(p\pi\xi)\,d\xi \sum_{r=0}^{\infty} (-)^r \phi\{p\pi(r+\xi)\}.$$

As the series is uniformly convergent we may integrate term by term, and we obtain

$$p\pi \sum_{r=0}^{\infty} \int_0^1 F\{p\pi(\xi+r)\}\,\phi\{p\pi(\xi+r)\}\,d\xi$$

$$= p\pi \sum_{r=0}^{\infty} \int_r^{r+1} F(p\pi t)\,\phi(p\pi t)\,dt$$

$$= p\pi \int_0^{\infty} F(p\pi t)\,\phi(p\pi t)\,dt$$

$$= \int_0^{\infty} F(x)\,\phi(x)\,dx.$$

The same argument would, of course, apply to the case in which $F(x) = \cos x\Theta(\sin^2 x)$.

§ 20. The argument of § 19 may be applied in many cases even when $F(x)$ is not continuous. Suppose for example

$$\Theta(\sin^2 x) = \frac{1}{\sqrt{(\sin^2 x)}},$$

so that

$$F(x) = \frac{\sin x}{\sqrt{(\sin^2 x)}} = (-1)^{[x/\pi]},$$

* See Hardy, *Proc. L.M.S.*, Ser. 2, Vol. IV., p. 250; Bromwich, *Infinite Series*, p. 114.

where $[y]$ denotes the algebraically greatest integer contained in y. Then we can prove, as in § 19, that, if p is a fixed odd integer,

$$\lim_{q\to\infty}(p\pi/q)\Sigma(-1)^{[np/q]}\phi(np\pi/q)=\int_0^\infty(-1)^{[x/\pi]}\phi(x)\,dx,$$

or, writing $\psi(x/\pi)$ for $\phi(x)$, and substituting y for x/π in the integral,

$$\lim_{q\to\infty}(p/q)\Sigma(-1)^{[np/q]}\psi(np/q)=\int_0^\infty(-1)^{[y]}\psi(y)\,dy,$$

p being odd.

If, in particular,

$$\psi(y)=\frac{1}{a+y}\qquad(a>0),$$

the value of the integral is

$$\int_0^1\frac{dy}{a+y}-\int_1^2\frac{dy}{a+y}+\int_2^3\frac{dy}{a+y}-\ldots$$

$$=\log\left\{\frac{a+1}{a}\cdot\frac{a+1}{a+2}\cdot\frac{a+3}{a+2}\cdot\frac{a+3}{a+4}\ldots\right\}.$$

It is easy to verify that this is the limit of the series. For if $n=rq+\mu$ $(0\leqq\mu<q)$, p being odd, we have

$$(-1)^{[np/q]}=(-1)^{rp+[\mu p/q]}=(-1)^{r+[\mu p/q]}.$$

Thus, writing $\Sigma_0^\infty(-1)^r/(x+r)=\beta(x)$, we find

$$S=\frac{p}{q}\Sigma\frac{(-1)^{[np/q]}}{a+(rq+\mu)(p/q)}=\frac{p}{q}\sum_{\mu=0}^{q-1}(-1)^{[\mu p/q]}\frac{1}{p}\beta\left\{\frac{a+(\mu p/q)}{p}\right\}.$$

Now if q is of the form $pk+1$, the sign of $(-1)^{[\mu p/q]}$ is positive when μ ranges from 0 to k, negative from $k+1$ to $2k$, and generally is the same as that of $(-1)^\nu$ from $\nu k+1$ to $(\nu+1)k$; thus we have

$$S=\sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{p}\sum_{\mu=\nu k+1}^{(\nu+1)k}\frac{p}{q}\beta\left\{\frac{a+(\mu p/q)}{p}\right\}.$$

But, by the ordinary definition of a definite integral,

$$\lim_{k\to\infty}\sum_{\nu k+1}^{(\nu+1)k}\frac{p}{q}\beta\left\{\frac{a+(\mu p/q)}{p}\right\}=\int_\nu^{\nu+1}\beta\left(\frac{a+t}{p}\right)dt=\int_0^1\beta\left(\frac{a+\nu+t}{p}\right)dt.$$

Thus

$$\lim_{k\to\infty} S=\sum_{\nu=0}^{p-1}\frac{(-1)^\nu}{p}\int_0^1\beta\left(\frac{a+\nu+t}{p}\right)dt=\int_0^1\beta(a+t)\,dt,$$

this transformation following from the fact that p is odd*. On evaluating the last integral, we obtain

$$\lim_{k\to\infty} S=\log\frac{\Gamma(\frac{1}{2}a)\,\Gamma\{\frac{1}{2}(a+2)\}}{[\Gamma\{\frac{1}{2}(a+1)\}]^2}*;$$

and this agrees with the value of $\int_0^\infty\frac{(-1)^{[y]}}{a+y}\,dy$ found above.

If q is of the form $pk+l$ $(1<l<p)$, it is not easy to specify briefly the values of μ at which the sign of $(-1)^{[\mu p/q]}$ changes; but the slight irregularities so introduced into the formula for S can have no effect on the final limit, since each group of terms is either k or $k+1$ in number, and the displacement of the beginning and end of the groups can never be more than l places.

It is not difficult to find other sequences of values of h yielding the same limit. But the series

$$\Sigma(-1)^{[n\eta]}/(a+n\eta)$$

is not convergent, if $\eta=h/\pi=p/q$, and p is even and q odd, so that there can be no question of its having a limit as $h\to 0$.

The proof of the last statement follows at once from the fact that, if $n=rq+\mu$, we have now

$$(-1)^{[np/q]}=(-1)^{[\mu p/q]},$$

so that the signs in the series are repeated in groups each containing q terms; suppose that there are κ positive and λ negative terms in each group. Then in the whole series the limit of the ratio of the number of positive to the number of negative terms is κ/λ; and this ratio cannot be unity, since $\kappa+\lambda=q$ is odd. Hence, by a theorem due to Cesàro†, the series cannot converge.‡

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* Nielsen, *Gamma-Funcktion*, pp. 16, 19. † Bromwich, *Infinite Series*, p. 53.

‡ It must be noticed, however, that this result does not imply the divergence of the series of § 19, where Θ is continuous. It appears that for these special values of $h=p\pi/q$ (p even and q odd), the condition (2) of § 13 is satisfied by the sine-series, which therefore converges, and tends (as $q\to\infty$) to the integral as a limit; but the cosine-series can be made to diverge, for any value of q, by suitable adjustment of Θ.

ADDITION TO PAPERS ON THE EQUATIONS OF MECHANICS.

By Philip E. B. Jourdain.

THIS paper contains, firstly, a correction (§§ I., II.) of some formulæ in a paper of mine on the equations of mechanics which appeared in the *Quarterly Journal*, Vol. xxxvi., 1905, pp. 284–296, and, secondly (§§ III., IV.), some further remarks, supplementary to those in my paper in the same *Journal*, Vol. xxxvi., 1904, pp. 61–79, on those principles of mechanics which consist in the equating of the variations of certain integrals to zero.

I.

Some of the results in my paper on "Alternative Forms of the Equations of Mechanics*" are rendered incorrect by the slip in calculation involved in putting

$$\frac{\partial x''}{\partial \dot{q}'} = \frac{\partial x'}{\partial q}.$$

In fact, we have for holonomous systems,

$$x' = \sum_\nu \frac{\partial x}{\partial q_\nu} q_\nu' + \frac{\partial x}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$x'' = \sum_\nu \left(q_\nu'' \frac{\partial x}{\partial q_\nu} + q_\nu' \sum_\mu \frac{\partial^2 x}{\partial q_\mu \partial q_\nu} q_\mu' + 2 q_\nu' \frac{\partial^2 x}{\partial t \partial q_\nu} \right) + \frac{\partial^2 x}{\partial t^2} \ldots (2),$$

and hence

$$\frac{\partial x''}{\partial \dot{q}_\bullet'} = 2 \sum_\mu \frac{\partial^2 x}{\partial q_\mu \partial q_\bullet} q_\mu' + 2 \frac{\partial^2 x}{\partial t \partial q_\bullet} = 2 \frac{\partial x'}{\partial q_\bullet} \quad \ldots\ldots\ldots (3).$$

More generally, if the system is not holonomous, that is to say, if there is an equation for x' which is not integrable, so that

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* *Quarterly Journal of Mathematics*, Vol. xxxvi., 1905, pp. 284–296. In equation (4) on p. 285, and on line 1 of p. 286, *for* $\frac{\partial x_r}{\partial q_\nu'}$, *read* $\frac{\partial x_{r'}}{\partial q_\nu'}$.

we cannot use $\dfrac{\partial x}{\partial q_\nu}$ or $\dfrac{\partial x}{\partial t}$ for coefficients in the given equation

$$x' = \sum_\nu A_\nu . q_\nu' + B,$$

or

$$x' = \sum \frac{\partial x'}{\partial q_\nu'} q_\nu' + B,$$

which now corresponds to (1), we can only write (unless we introduce B', which we do not want), instead of (2),

$$x'' = \sum_\nu \frac{\partial x'}{\partial q_\nu'} q_\nu'' + \sum_\nu \frac{\partial x'}{\partial q_\nu} q_\nu' + \frac{\partial x'}{\partial t}.$$

Hence

$$\frac{\partial x''}{\partial q_s'} = \sum_\nu q_\nu'' \frac{\partial^2 x'}{\partial q_s' \partial q_\nu'} + \frac{\partial x'}{\partial q_s} + \sum_\nu q_\nu' \frac{\partial^2 x'}{\partial q_s' \partial q_\nu} + \frac{\partial^2 x'}{\partial q_s' \partial t}$$

$$= \frac{\partial x'}{\partial q_s} + \frac{d}{dt} \frac{\partial x'}{\partial q_s'} * \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

This result (4) includes (3), for if, in (4), the system is holonomous,

$$\frac{\partial x'}{\partial q_s'} = \frac{\partial x}{\partial q_s}, \text{ so that } \frac{d}{dt} \frac{\partial x'}{\partial q_s'} = \frac{\partial x'}{\partial q_s} \dagger.$$

When‡ $$W = x'x'', \quad W_1 = x'^2,$$

we have, if the system be holonomous,

$$\frac{\partial W}{\partial q'} - \frac{\partial W_1}{\partial q} = x'' \frac{\partial x'}{\partial q'}$$

as

$$\frac{\partial x''}{\partial q'} = 2 \frac{\partial x'}{\partial q}.$$

---

* From this, also, we get

$$\frac{d}{dt} \frac{\partial x'}{\partial q_s'} = \frac{\partial x''}{\partial q_s'} - \frac{\partial x'}{\partial q_s},$$

so that (see *Math. Gaz.*, Vol. II., 1903, p. 339)

$$\frac{\partial \mathfrak{T}}{\partial q_\nu''} = \frac{d}{dt}\left(\frac{\partial T}{\partial q_\nu'}\right) - \sum_r m_r \left(x_r' \frac{\partial x_r''}{\partial q_\nu'} + \ldots\right) + \frac{\partial T}{\partial q_\nu}.$$

† This should take the place of line 3 on p. 286 of my paper in the *Quarterly Journal of Mathematics*, Vol. XXXVI.

‡ Cf., *ibid.*, pp. 286–287.

Thus, for such a system, if

$$W = T', \quad W_1 = 2T,$$

an alternative form of Lagrange's equations is

$$\frac{\partial T'}{\partial q_\nu'} - 2\frac{\partial T}{\partial q_\nu} = Q_\nu \quad (\nu = 1, 2, \ldots);$$

a form at once verified, since

$$\frac{\partial T'}{\partial q'} = \frac{d}{dt}\left(\frac{\partial T'}{\partial q'}\right) + \frac{\partial T}{\partial q}.$$

The errors in the table on p, 288 can easily be found, and we shall now restrict our attention to the most important case*, where

$$W = \tfrac{1}{2}\Sigma m_r (x_r x_r'' + \ldots) = \tfrac{1}{2} Y'' - T,$$

where $$Y = \tfrac{1}{2}\Sigma m_r (x_r^2 + \ldots),$$

and the system is holonomous. Here our equations of motion may be written†

$$\left(\frac{d}{dt}\right)^2 \frac{\partial W}{\partial q_\nu''} - \left(\frac{d}{dt}\right)\frac{\partial W}{\partial q_\nu'} + \frac{\partial W}{\partial q_\nu} = Q_\nu (\nu = 1, 2, \ldots),$$

the left-hand side of which may be written

$$\tfrac{1}{2}\left(\frac{d}{dt}\right)^2 \frac{\partial Y''}{\partial q_\nu''} - \tfrac{1}{2}\left(\frac{d}{dt}\right)\frac{\partial Y''}{\partial q_\nu'} + \left(\frac{d}{dt}\right)\frac{\partial T}{\partial q_\nu'} + \tfrac{1}{2}\frac{\partial Y''}{\partial q_\nu} - \frac{\partial T}{\partial q_\nu},$$

and this reduces, because

$$\frac{\partial Y''}{\partial q''} = \frac{\partial Y}{\partial q},$$

$$\frac{\partial Y''}{\partial q'} = \Sigma m_r \left(x_r \frac{\partial r_r''}{\partial q'} + 2x_r' \frac{\partial x_r'}{\partial q'}\right) + 2\frac{d}{dt}\frac{\partial Y}{\partial q},$$

$$\frac{\partial Y''}{\partial q} = \frac{d}{dt}\frac{\partial Y'}{\partial q} = \left(\frac{d}{dt}\right)^2 \frac{\partial Y}{\partial q},$$

to the Lagrangian form.

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* *Ibid.*, pp. 289, seq.

† Omit the 2 in equation (9), and the preceding one, on p. 289.

II.

Where δ denotes a variation which affects t also, and accordingly

$$\delta x' = \frac{d\delta x}{dt} - x' \frac{d\delta t}{dt},$$

$$\delta x'' = \frac{d^2\delta x}{dt^2} - 2x'' \frac{d\delta t}{dt} - x' \frac{d^2\delta t}{dt^2},$$

we have $2\delta W = \Sigma_r m_r \{(x_r'' \delta x_r + x_r \delta x_r'') + \dots\}$

$$= \Sigma_r m_r \left[\left\{ x_r'' \Sigma_\nu \frac{\partial x_r}{\partial q_\nu} \delta q_\nu + x_r'' \frac{\partial x_r}{\partial t} \delta t + x_r \left(\frac{d}{dt}\right)^2 \left(\Sigma_\nu \frac{\partial x_r}{\partial q_\nu} \delta q_\nu + \frac{\partial r_r}{\partial t} \delta t \right) \right.\right.$$

$$\left.\left. - 2x_r x_r'' \frac{d\delta t}{dt} - x_r x_r' \frac{d^2\delta t}{dt^2} \right\} + \dots \right].$$

Substituting for x'' and x' in the last two terms in $\{\ \}$ from the equalities

$$x' = \Sigma_\nu \frac{\partial x}{\partial q_\nu} q_\nu' + \frac{\partial x}{\partial t},$$

$$x'' = \Sigma_\nu \frac{\partial x}{\partial q_\nu} q_\nu'' + \Sigma_\nu \frac{\partial x'}{\partial q_\nu} q_\nu' + \frac{\partial x'}{\partial t},$$

we get

$$2\delta W = \Sigma_r m_r \left[\left\{ \left(x_r'' \frac{\partial r_r}{\partial t} + x_r \frac{\partial x_r''}{\partial t} \right) \delta t + \Sigma_\nu \left(x_r'' \frac{\partial x_r}{\partial q_\nu} + x_r \frac{\partial x_r''}{\partial q_\nu} \right) \delta q_\nu \right.\right.$$

$$+ 2\Sigma_\nu \left(x_r \frac{\partial x_r'}{\partial q_\nu} \frac{d\delta q_\nu}{dt} - x_r \frac{\partial x_r'}{\partial q_\nu} q_\nu' \frac{d\delta t}{dt} \right)$$

$$\left.\left. + \Sigma_\nu \left(x_r \frac{\partial x_r}{\partial q_\nu} \frac{d^2\delta q_\nu}{dt^2} - 2x_r \frac{\partial x_r}{\partial q_\nu} q_\nu'' \frac{d\delta t}{dt} - x_r \frac{\partial x_r}{\partial q_\nu} q_\nu' \frac{d^2\delta t}{dt^2} \right) \right\} + \dots \right],$$

and hence

$$\delta W = \frac{\partial W}{\partial t} \delta t + \Sigma_\nu \left(\frac{\partial W}{\partial q_\nu} \delta q_\nu + \frac{\partial W}{\partial q_\nu'} \delta q_\nu' + \frac{\partial W}{\partial q_\nu''} \delta q_\nu'' \right),$$

and *not** $2 \dfrac{\partial W}{\partial q_\nu'}$ as coefficient of $\delta q_\nu'$, since

$$\frac{\partial x_r''}{\partial q_\nu'} = 2 \frac{\partial x_r'}{\partial q_\nu'}.$$

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* The opposite of this was stated on p. 291 of my above-cited paper (cf. also *Math. Ann.*, Bd. LXII., 1905, p. 416, note).

When, however, the function W is formed for a position on a neighbouring 'path' of the system, this position being obtained by giving every rectangular coordinate of the system *the* virtual* displacement,

$$\delta_1 x = \sum_\nu \frac{\partial x}{\partial q_\nu}\,\delta q_\nu,$$

W becomes altered by a quantity which may be *called* a 'variation' of W only provided that we do not use the word literally; that is to say provided that we do not require of a 'variation' of W (δW), that we have

$$\delta W = \frac{\partial W}{\partial t}\,\delta t + \sum_\nu \left(\frac{\partial W}{\partial q_\nu}\,\delta q_\nu + \frac{\partial W}{\partial q_\nu'}\,\delta q_\nu' + \frac{\partial W}{\partial q_\nu''}\,\delta q_\nu''\right).$$

It is better to use† a new word, say 'alteration' (δ_1); and we then have the following definition. If δ stands for any variation which affects the geometrical coordinates and t, we define an 'alteration' (δ_1) of any function (F) of the x's, y's, and z's and their derivatives to be that quantity that arises from δF when the δx's, δy's, and δz's are all replaced by expressions for *the above virtual* displacements. In our case,

$$\delta x = \frac{\partial x}{\partial t}\,\delta t + \sum_\nu \frac{\partial x}{\partial q_\nu}\,\delta q_\nu$$

is to be replaced by

$$\delta_1 x = \sum_\nu \frac{\partial x}{\partial q_\nu}\,\delta q_\nu = \delta x - \frac{\partial x}{\partial t}\,\delta t.$$

Thus we have

$$2\delta_1 W = \sum_r m_r \left[\left\{x_r'' \sum_r \frac{\partial x_r}{\partial q_\nu}\,\delta q_\nu + x_r \left(\frac{d}{dt}\right)^2 \left(\sum_\nu \frac{\partial x_r}{\partial q_\nu}\,\delta q_\nu\right) - 2x_r x_r'' \frac{d\delta t}{dt} - x_r x_r' \frac{d^2\delta t}{dt^2}\right\} + \dots\right]$$

$$= \sum_r m_r \left[\left\{\sum_\nu \left(x_r'' \frac{\partial x_r}{\partial q_\nu} + x_r \frac{\partial x_r''}{\partial q_\nu}\right)\delta q_\nu + 2\sum_\nu x_r \frac{\partial x_r'}{\partial q_\nu}\,\frac{d\delta q_\nu}{dt} + \sum_\nu x_r \frac{\partial x_r}{\partial q_\nu}\,\frac{d^2\delta q_\nu}{dt^2}\right\} + \dots\right] - 4W\frac{d\delta t}{dt} - Y'\frac{d^2\delta t}{dt^2}$$

* Another kind (Routh's) of virtual displacement is also possible (see below).

† After the example of Hölder, who said "Abänderung" (see *Quarterly Journal of Mathematics*, Vol XXXVI., p. 75, note).

$$= 2\Sigma_\nu \left(\frac{\partial W}{\partial q_\nu} \delta q_\nu + \frac{\partial W}{\partial q_\nu'} \frac{d\delta q_\nu}{dt} + \frac{\partial W}{\partial q_\nu''} \frac{d^2 \delta q_\nu}{dt^2} \right) - 4W \frac{d\delta t}{dt} - Y' \frac{d^2 \delta t}{dt^2}.*$$

Hence the vanishing of the integral

$$\int_{t_0}^{t_1} \left(\delta_1 W - \delta' U + 2W \frac{d\delta t}{dt} + \tfrac{1}{2} Y' \frac{d^2 \delta t}{dt^2} \right) dt$$

is necessary and sufficient that Lagrange's equations subsist for the mechanical system, provided that the δq's and the $\frac{d\delta q}{dt}$'s vanish at the limits of integration.†

III.

Now, it appears at first sight that this discovery of integral-principles other than the one

$$\int_{t_0}^{t_1} \left(\delta_1 T + 2T \frac{d\delta t}{dt} + \delta' U \right) dt = 0 \ldots\ldots\ldots\ldots (5),$$

which is a generalisation of the principles of Hamilton and of Least Action, invalidates my previous assertion‡ that (5) is the most general integral-variational-principle. But (5) *is* the most general principle obtained from the differentation of the *single function* T (when we abstract from the term $\delta' U$ representing the external forces); when we admit other functions (such as Y and W), we get other principles, and the considerations in my paper of 1905 show how all such principles can be found.

But there is another question relating to the variational principles. We have formulated after Hölder the principles

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* This may also be written

$$2\Sigma_\nu \left(\frac{\partial W}{\partial q_\nu} \delta q_\nu + \frac{\partial W}{\partial q_\nu'} \delta q_\nu' + \frac{\partial W}{\partial q_\nu''} \delta q_\nu'' \right) - 2\Sigma_r m_r \left(2x_r' \frac{\partial x_r'}{\partial t} \frac{d\delta t}{dt} + x_r \frac{\partial x_r}{\partial t} \frac{d^2 \delta t}{dt^2} \right)$$

$$= 2\delta W - 2\Sigma_r m_r \left\{ x + \left(\frac{d}{dt} \right)^2 \left(\frac{\partial x_r}{\partial t} \delta t \right) + x_r'' \frac{\partial x_r}{\partial t} \delta t \right\}.$$

† The requirement that the $\frac{d\delta q}{dt}$'s vanish at the limits is *not* equivalent to the requirement that the $d\delta q$'s vanish at the limits (cf. my above-cited paper, pp. 292–293), unless δt always $= 0$, which makes the 'varied path' a 'Hamilton's path,' and reduces the above integral principle to the simple form

$$\int_{t_0}^{t_1} (\delta_1 W - \delta' U) = 0.$$

‡ *Quarterly Journal of Mathematics*, Vol. XXXVI., p. 78.

with a conception of 'alteration,' allied, but not identical, with the strict conception of 'variation.' And so the problem of discovering an expression (if there be one) whose variation equated to zero is equivalent to the system of Lagrange's equations is a different one. This question was naturally connected with the principle of Least Action, since $\delta' U$ is only a variation if a force-function exists, and only δT occurs under the integral in the principle of Least Action. We will next suppose, for simplicity, that such a U exists.

Our first problem is: Is there a function

$$\Phi(q_1, q_2, \ldots, q_n, q_1', \ldots, q_n', t)$$

such that, where δ is a variation affecting the q's, q''s, and t, the equation

$$\delta \int_{t_0}^{t_1} \Phi . dt = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

is equivalent to the system

$$\frac{d}{dt}\frac{\partial L}{\partial q_\nu'} - \frac{\partial L}{\partial q_\nu} = 0 \quad (\nu = 1, 2, \ldots, n),$$

where $L = T + U$, and U is explicity independent of the q''s?

Our (6) becomes*

$$\left| \sum_\nu \frac{\partial \Phi}{\partial q_\nu'} \delta q_\nu \right|_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \sum_\nu \left(\frac{\partial \Phi}{\partial q_\nu} - \frac{d}{dt}\frac{\partial \Phi}{\partial q_\nu'} \right) \delta q_\nu$$

$$+ \int_{t_0}^{t_1} dt \left(\Phi \frac{d\delta t}{dt} + \frac{\partial \Phi}{\partial t} \delta t - \sum_\nu q_\nu' \frac{\partial \Phi}{\partial q_\nu'} \frac{d\delta t}{dt} \right) = 0,$$

or, integrating by parts, to get rid of the $d\delta t$'s,

$$\left| \sum_\nu \frac{\partial \Phi}{\partial q_\nu'} (\delta q_\nu - q_\nu' . \delta t) + \Phi . \delta t \right|_{t_0}^{t_1}$$

$$+ \int_{t_0}^{t_1} dt \sum_\nu \left(\frac{\partial \Phi}{\partial q_\nu} - \frac{d}{dt}\frac{\partial \Phi}{\partial q_\nu'} \right) (\delta q_\nu - q_\nu' . \delta t) = 0.$$

This form of the equation, when we put $\Phi = L$, was given

* Here we get certain terms (which we suppose to vanish) outside the integral-sign. Voss (*Münch. Ber.*, 1901, pp. 171–177) has used Hölder's generalised 'variations' to formulate very general integral-principles, *without* this integration by parts.

by Routh*. He seems to have remarked that the equation $\delta\int_{t_0}^{t_1}\Phi.dt=0$ is equivalent to the system of Lagrange's equations in two (at least) cases:

(1) If $\Phi=T+U$, δt is always zero, and the δq's vanish at the limits of integration (Hamilton's principle).†

(2) If $\Phi=T+U+h$, where h is a constant, and one such that $\delta h=0$, the δq's vanish at the limits, and, δt not vanishing, the system is conservative and such that

$$T-U=h.$$

Then we have

$$\delta\int_{t_0}^{t_1}2T.dt\left[\text{or its equivalent}\int_{t_0}^{t_1}\Sigma_\nu\left(\frac{\partial L}{\partial q_\nu}-\frac{d}{dt}\frac{\partial L}{\partial q_\nu'}\right)(\delta q_\nu-q_\nu'.\delta t)\right]=0,$$

where δ represents a variation affecting t also, and in which the energy, on variation of the mechanical system from its position P (at time t) to its corresponding position ($P+\delta P$, at time $t+\delta t$) in the neighbouring motion, is constant, or

$$\delta(T-U)=0.‡$$

IV.

There is one point in Routh's discussion which must be referred to here in some detail, because of a misunderstanding to which it has given rise.§

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* "A Treatise on the Dynamics of a System of Rigid Bodies," Part II (5th ed.), 1892, § 442 seq. These investigations first appeared (in Routh's book) in 1877.

† Routh (*op. cit*, p. 281) claimed the theorem that $\delta\int_{t_0}^{t_1}L.dt=0$, when the δq's vanish at the limits and the time of transit is given ($\delta t_1=\delta t_0$), as first given in his treatise in 1877. On this claim two remarks are to be made. (1) This ('Hamilton's') principle has been long previously given (see note on p. 90 of the *Encykl. der Math. Wiss.* IV., 1): (2) Routh's condition $\delta t_1=\delta t_0$ is not sufficient for the purpose; we require $\delta t=0$ *in general if T or L contain t explicitly*, for only then does

$$\left|\left(T+U-\Sigma_\nu q_\nu'\frac{\partial T}{\partial q_\nu'}\right)\delta t\right|_{t_0}^{t_1}=0,$$

unless, indeed,

$$U=T+\Sigma_r m_r\left(x_r'\frac{\delta x_r}{\delta t}+\ldots\right)$$

(see *Quarterly Journal of Mathematics*, Vol. XXXVI., p. 77, note).

Jacobi noticed that Hamilton's principle holds, even when U contains t explicitly.

‡ Notice that, in Routh's formulation, the principle of least action only holds for conservative systems.

§ Cf. *Quarterly Journal of Mathematics*, Vol. XXXVI., p. 75, note; *Math. Ann.*, Bd. LXII, 1905, pp. 417–418; and *ibid.*, Bd. LXIV., pp. 158–159.

Where $$S = \int_{t_0}^{t_1} L \, dt,$$

Routh, evaluating δS expressed in *rectangular* coordinates, obtained,* since T is a homogeneous quadratic function of the x''s,

$$\delta S = \Big| (U - T)\,\delta t + \sum_r m_r x_r' \delta x_r \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum_r \left(\frac{\partial U}{\partial x_r} - m_r x_r'' \right) (\delta x_r - x_r' \delta t)\, dt,$$

where the δx's are connected by the equations of condition. He then stated that, since $\delta x - x' \delta t$ is the projection on the x-axis of a displacement of m from its position in the actual motion at the time t to its position in the neighbouring motion *at the same time*, the part under the integral vanishes by the principle of virtual velocities †

This can only mean that $\delta x - x' \delta t$ is a virtual displacement; we shall now show that this is, in spite of the apparent contradiction with the fact (see above) that $\delta x - \dfrac{\partial x}{\partial t} \delta t$ is virtual, the case.

If the mechanical system is holonomous—and we need not consider the case of non-holonomy when we are developing the variation of our integral‡—we can express the $3n$ rectangular coordinates (x_r) of the system as functions of k independent (and accordingly of not more or not less than k) parameters (q_ν) and of t.§ Thus, if

$$x = \phi\,(q_1, q_2, \ldots, q_k, t),$$

we have

$$\delta x = \sum_\nu \frac{\partial \phi}{\partial q_\nu} \delta q_k + \frac{\partial \phi}{\partial t} \delta t.$$

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* Here we only write the x's of the $3n$ quantities x_r, y_r, z_r $(r = 1, 2, \ldots, n)$.

† Also, to interpret $\delta q - q' \delta t$, Routh remarked: q being the value of a coordinate in the actual motion at the time t, $q + \delta q$ is its value in the neighbouring motion of the time $t + \delta t$. But $q' \delta t$ is the change of q in the time δt; hence $q + \delta q - q' \delta t$ is the value of the coordinate in the neighbouring motion at the time t.

In Routh's extension to forces Q_ν, which have no force function, he remarked that the neighbouring motion must be such that

$$\sum_\nu Q_\nu (\delta q_\nu - q_\nu' \delta t) = 0$$

(cf. the text later on).

‡ See *Quarterly Journal of Mathematics*, Vol. XXXVI., pp. 74–75.

§ Of the geometrical quantities q_ν *alone*, in some cases.

But a virtual displacement (δx) of x at the time t—a displacement consistent with the equations of condition at that instant (which brings about that t is to be regarded as constant in calculating the value of this virtual displacement*)—is given by *both*

$$\delta_1 x = \Sigma_\nu \frac{\partial \phi}{\partial q_\nu} \delta q_\nu,$$

and hence

$$\delta_1 x = \delta x - \frac{\partial x}{\partial t} \delta t,$$

and

$$\delta x - x' \delta t = \Sigma \frac{\partial \phi}{\partial q_\nu} (\delta q_\nu - q_\nu' \delta t).$$

In each case of the latter we merely use different displacements ($\delta q - q' \delta t$) of the q's from those used (δq) in the corresponding case of the former.

On the difference in concept between the δ_1-process of Hölder (used by myself), and the virtual variations of Routh, Voss (1900), and Réthy, I have dwelt very fully in a paper shortly to appear in the *Mathematische Annalen*.

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BIRATIONAL TRANFORMATION OF SURFACES.

By A. B. Basset, M.A., F.R.S.

1. In a recent paper I employed the theory of birational transformation to investigate the constituents of the point, line, and mixed singularities of plane curves; and in a subsequent one I investigated by a different method the point constituents of various kinds of multiple points on surfaces. The transformation which I employed was the quadric one; and in the present paper I shall show that the same transformation may be used to analyse the point constituents of various kinds of multiple points on surfaces. One result of the investigation is to show that there is an important analogy between the theory of curves and surfaces, and that a fairly complete theory exists with respect to the changes produced in the constituents of a multiple point on a surface, when the tangent cone possesses singular generators.

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* This was noticed by Lagrange, who completely anticipated Vieille in showing that the equations of condition may contain t explicitly, without his calculation being affected (see my note in *Bibl. Math.*, 3. Folge, Bd. VI., 1906, pp. 350–353).

The letters S.C., S.S., and M.P. will be employed as abbreviations of my papers on *Compound Singularities of Curves*,* *Singularities of Surfaces*,† and *Multiple Points on Surfaces*.‡

2. Let P be any point whose coordinates are $(\xi, \eta, \zeta, \omega)$; let $\Omega = (\beta, \gamma, \delta)^2$; and let AP cut the polar plane of P with respect to the quadric $\alpha^2 = \Omega$ in P'. Then the equation of the polar plane of P is

$$2\alpha\xi - \beta d\Omega_1/d\eta - \gamma d\Omega_1/d\zeta - \delta d\Omega_1/d\omega = 0 \ldots\ldots(1),$$

where $\Omega_1 = (\eta, \zeta, \omega)^2$; and those of AP are

$$\beta/\eta = \gamma/\zeta = \delta/\omega = h \text{ (say)} \ldots\ldots\ldots\ldots(2);$$

whence, if $(\xi', \eta', \zeta', \omega')$ be the coordinates of P', we obtain from (1) and (2)

$$2\xi\xi' = h(\eta d\Omega_1/d\eta + \zeta d\Omega_1/d\zeta + \omega d\Omega_1/d\omega) = 2h\Omega_1 \ldots(3),$$

accordingly, by (2) and (3),

$$\frac{\xi\xi'}{\Omega_1} = \frac{\eta'}{\eta} = \frac{\zeta'}{\zeta} = \frac{\omega'}{\omega}.$$

From this it follows that if $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ be the coordinates of a pair of corresponding points P and P', these quantities are connected by the equations

$$\frac{\alpha'}{\Omega} = \frac{\beta'}{\alpha\beta} = \frac{\gamma'}{\alpha\gamma} = \frac{\delta'}{\alpha\delta} = k \text{ (say)} \ldots\ldots\ldots(4).$$

Substituting the values of β', γ', δ' from the last three of (4), we obtain

$$\Omega' = k^3\alpha^n\Omega = \alpha^2\alpha'^2/\Omega,$$

whence

$$\frac{\alpha}{\Omega'} = \frac{\beta}{\alpha'\beta'} = \frac{\gamma}{\alpha'\gamma'} = \frac{\delta}{\alpha'\delta'} \ldots\ldots\ldots\ldots(5).$$

The value of Ω will usually be written in the form

$$\Omega = \beta^2 + \beta\Omega_1 + \Omega_2 \ldots\ldots\ldots\ldots(6),$$

where $\Omega_n = (\gamma, \delta)^n$.

3. *A conic node, whose position is arbitrary, gives rise to a conic node on the transformed surface.*

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* Vol. XXXVII., p. 313. † Vol. XXXVIII., pp. 63 and 159. ‡ Vol. XXXIX., p. 1.

To prove this it will be sufficient to consider the quartic surface

$$\alpha^2(\beta^2+\beta v_1+v_2)+\alpha(2\lambda\beta^3+\beta^2 w_1+\beta w_2+w_3)$$
$$+\lambda^2\beta^4+(\lambda w_1-\lambda^2 v_1)\beta^3+\beta^2 V_2+\beta V_3+V_4=0,$$

which has a conic node at A and also at the point B', where AB is cut by the plane $\alpha+\lambda\beta=0$. The transformed equation may be put into the form

$$(\Omega+\lambda\alpha\beta)^2\beta^2+(\Omega^2-\lambda^2\alpha^2\beta^2)\beta v_1+(\Omega+\lambda\alpha\beta)\alpha\beta^2 w_1$$
$$+\alpha^2(\beta^2 V_2+\beta V_3+V_4)+\alpha\Omega(\beta w_2+w_3)+\Omega^2 v_2=0.$$

This is the equation of a sextic surface which has a quadruple point at A, and a conic node at the point B', where AB is cut by the plane $\lambda\alpha+\beta=0$. The quadric $\Omega+\lambda\alpha\beta=0$, in which the plane $\alpha+\lambda\beta=0$ is transformed, also passes through B'.

If the conic node on the original surface is situated at B, $\lambda=0$; whence the conic node on the transformed surface is situated at A. Hence: *Every conic node situated at a point B in the plane α becomes a conic node at A on the transformed surface, which moves up to coincidence with A along the line AB.*

4. We are now in a position to apply the theory of birational transformation to multiple points on surfaces.

(i) *A multiple point of order p on a surface corresponds to a curve of degree p on the transformed surface, which is identical with the section of the tangent cone by the plane α.*

The equation of the original surface is of the form

$$\alpha^{n-p}u_p+\alpha^{n-p-1}u_{p+1}+\alpha^{n-p-2}u_{p+2}+\dots u_n=0 \quad \dots\dots(7),$$

and that of the transformed surface is

$$\alpha^{n-p}u_n+\alpha^{n-p-1}\Omega u_{n-1}+\dots \alpha^2\Omega^{n-p-2}u_{p+2}+\alpha\Omega^{n-p-1}u_{p+1}+\Omega^{n-p}u_p=0$$
$$\dots\dots(8).$$

The latter surface is therefore of degree $2n-p$, and has a multiple point of order n at A; also the section of (8) by the plane α is a multiple conic of order $n-p$ and a curve of degree p.

(ii) *Every nodal generator of the tangent cone at A, which is not a line of closest contact, corresponds to a node on the section of the transformed surface by the plane α.*

Let

$$\left.\begin{aligned} u_p &= \beta^{p-2}v_2 + \beta^{p-3}v_3 + \dots v_p \\ u_{p+1} &= \beta^{p+1}w_0 + \beta^p w_1 + \dots w_{p+1} \\ u_{p+2} &= \beta^{p+2}W^2 + \beta^{p+1}W_1 + \dots w_{p+2} \end{aligned}\right\} \quad \dots\dots\dots(9),$$

then AB is a nodal generator of the cone u_p. Also the highest power of β in (8) is the $(2n-p-1)^{\text{th}}$, and its coefficient is αw_0, which shows that the plane α touches the transformed surface at B; hence B is a node on the section.

It follows from M.P., § 3, that every nodal generator of the tangent cone at A converts one of the constituent conic nodes into a binode; and the preceding result shows that, if the plane α touches a surface at δ points, the transformed surface has a multiple point, the tangent cone at which has δ nodal generators.

(iii) *Every cuspidal generator of the tangent cone at A, which is not a line of closest contact, corresponds to a cusp on the section of the transformed surface by the plane α.*

This may be proved in a similar manner by putting $v_2 = v_1^2$. In the same way it can be shown that if the tangent cone at A has a singular generator of any species whatever, the section of the transformed surface by the plane α possesses a singular point of the same character.

We shall now suppose that the singular generator of the tangent cone at A is a line of closest contact, in which case $w_0 = 0$.

(iv) *Every nodal generator of the tangent cone at A, which is a line of closest contact, corresponds to a conic node on the section of the transformed surface by the plane α; and such a generator adds an additional conic node to the constituents of the multiple point.*

When $w_0 = 0$, the highest power of β is the $(2n-p-2)^{\text{th}}$, and its coefficient is

$$\alpha^2 W^2 + \alpha w_1 + v_2 \quad \dots\dots\dots(10),$$

which shows that B is a conic node on the transformed surface. Also we have shown in § 3 that this conic node becomes a conic node on the original surface, which moves up to coincidence with A along the line AB. This furnishes another proof of the theorem S.S., § 13.

(v) *Every cuspidal generator of the tangent cone at A, which is a line of closest contact, corresponds to a conic node on the*

section of the transformed surface such that the tangent cone thereat touches the plane α; and such a generator adds an additional binode to the constituents of the multiple point.

In this case $v_2 = v_1^2$, and the expression (10) equated to zero gives the tangent cone at the point B on the transformed surface. It follows from § 3, or from S.S., § 13, that an additional double point is added to the multiple point at A on the original surface, and M.P., § 6, (i), shows that this double point is a binode.

The results (iii) and (v) fail when $p = 2$; for in this case the cone u_p become a pair of coincident planes and the singularity at A on the original curve is a unode.

(vi) *Every binode on the section of the transformed surface by the plane α adds a binode to the constituents of the multiple point on the original surface.*

Let $w_1 = W(v_1 + \tau_1)$, $v_2 = v_1\tau_1$, then the expression (10) becomes

$$(\alpha W + v_1)(\alpha W + \tau_1),$$

which shows that the singularity at B on the transformed surface is a binode whose axis is arbitrary. The original surface may now be written in the form

$$\alpha^{n-p-2}\beta^{p-2}(\alpha v_1 + W\beta^2)(\alpha\tau_1 + W\beta^2) + \alpha^{n-p}(\beta^{p-3}v_3 + \dots v_p)$$
$$+ \alpha^{n-p-1}(\beta^{p-1}w_2 + \dots w_{p+1}) + \alpha^{n-p-2}(\beta^{p+1}W_1 + \dots W_{p+2})$$
$$+ \alpha^{n-p-3}u_{p+3} + \dots u_n = 0 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (11).$$

By means of the methods explained in S.C., §§ 7 to 12, it can be shown that the section of (11) by the plane $v_1 = \tau_1$ is a curve having a rhamphoid cusp at A and $p-2$ ordinary branches through it. To find the reduction of class, it will be sufficient to consider a quartic surface, since the method employed is applicable to any surface. Putting $n = 4$, $p = 2$, $v_1 = \delta$, $\tau_1 = p\gamma + q\delta$, the surface (11) becomes

$$(\alpha\delta + W\beta^2)\{\alpha(p\gamma + q\delta) + W\beta^2\} + \alpha(\beta w_2 + w_3)$$
$$+ \beta^3 W_1 + \beta^2 W_2 + \beta W_3 + W_4 = 0 \dots (12),$$

and the first polar of D, which may be any arbitrary point, is

$$\alpha\{\alpha(p\gamma + q\delta) + W\beta^2\} + q\alpha(\alpha\delta + W\beta^2) + \alpha(\beta w_2'' + w_3'')$$
$$+ \beta^3 W_1'' + \dots \dots W_4'' = 0 \dots (13).$$

The sections of (12) and (13) by the plane $\delta = p\gamma + q\delta$ may be written in the form

$$(\alpha\delta + W\beta^2)^2 + \alpha\delta^2 A_1 + \delta A_3 = 0 \quad \ldots\ldots\ldots(14),$$

and

$$\alpha(\alpha\delta + W\beta^2) + \alpha\delta B_1 + B_3 = 0 \quad \ldots\ldots\ldots(15),$$

where $A_n \equiv B_n \equiv (\beta, \delta)^n$.

Eliminating α between (14) and (15), it will be found that the eliminant is a binary septimic of (β, δ), which shows that (14) and (15) have quinquetactic contact with one another at A. Accordingly the surface (12) and its first polar with respect to any arbitrary point intersect one another in five coincident points at A; hence the reduction of class is 5, which shows that the singularity at A is formed by the union of a conic and a binode.

The case of a cubic surface is peculiar, the explanation of which is given at the end of M.P., § 17. Let us consider the equation

$$\alpha u_2 + u_3 = 0 \quad \ldots\ldots\ldots(16),$$

which represents a cubic surface having a conic node at A. The transformed surface, when written at full length, becomes

$$\alpha(\beta^3 w_0 + \beta^2 w_1 + \beta w_2 + w_3) + \Omega(\beta^2 v_0 + \beta v_1 + v_2) = 0 \ldots(17).$$

In order that (17) should have a conic node at B, it is necessary that $w_0 = v_0 = v_1 = 0$, in which case the equation of the nodal cone is

$$\alpha w_1 + v_2 = 0,$$

and, if B is a binode, $v_2 = w_1\tau_1$ and (16) becomes

$$\alpha w_1\tau_1 + \beta^2 w_1 + \beta w_2 + w_3 = 0 \quad \ldots\ldots\ldots(18),$$

and the singularity at A is Salmon's binode B_5, which, as pointed out in M.P., § 17, is a singularity of a different character to the singular point $C=1$, $B=1$ on a surface of higher degree than the third.

5. The preceding results enable us to develop an important analogy between the theory of the birational transformation of curves and surfaces.

Let ABC be the triangle of reference of the curve, and $ABCD$ the tetrahedron of reference of the surface; then the following correspondence exists between the different elements of a curve and a surface, which is shown in the accompanying table.

Original Curve.	Transformed Curve.	Original Surface.	Transformed Surface.
(i) A multiple point of order p at A. Tangents distinct.	p distinct points on BC.	A multiple point of order p at A. Tangent cone anautotomic.	An anautotomic curve of degree p in the plane BCD.
(ii) Do. having a pair of coincident tangents.	2 coincident and $p-2$ distinct points on BC.	Do. A nodal generator on the tangent cone.	A node on the section by the plane BCD.
(iii) Do. having 3 coincident tangents.	3 coincident and $p-3$ distinct points on BC.	Do. A cuspidal generator on the tangent cone.	A cusp on the section by the plane BCD.
(iv) Do. having $2r$ coincident tangents, $r>0$.	$2r$ coincident and $p-2r$ distinct points on BC.	Do. δ nodal generators on the tangent cone, which move up to coincidence along a continuous curve.	δ nodes on the section, which move up to coincidence along a continuous curve.
(v) Do. having a pair of tacnodal and $p-2$ ordinary branches.	A node and $p-2$ distinct points on BC.	Do. a nodal generator on the tangent cone, which is a line of closest contact.	A conic node on the section by the plane BCD.
(vi) Do. having one pair of tacnodal and $2r$ ordinary branches which coincide with the tacnodal branches, and $p-2r-2$ ordinary branches.	A node and $2r$ ordinary points, which coincide with the node; and $p-2r-2$ distinct points on BC.	Do. A nodal generator on the tangent cone, which is a line of closest contact; and δ ordinary nodal generators which move up to coincidence along a continuous curve.	A conic node and δ ordinary nodes on the section by the plane BCD, which move up to coincidence along a continuous curve.
(vii) Do. having s pairs of tacnodal branches, r of which are coincident, and $p-2s$ ordinary branches.	s nodes on BC, r of which are coincident; and $p-2s$ distinct points.	Do. δ nodal generators on the tangent cone, all of which are lines of closest contact, and of which r are coincident.	δ conic nodes on the section by the plane BCD, r of which are coincident.
(viii) Do. consisting of a pair of tacnodal branches and one coincident ordinary branch, and $p-3$ distinct ordinary branches.	A node, with BC as one of the nodal tangents; and $p-3$ distinct points on BC.	A cuspidal generator on the tangent cone, which is a line of closest contact.	A conic node on the section, whose tangent cone touches the plane BCD.
(ix) Do. consisting of a rhamphoid cusp and $p-2$ distinct ordinary branches passing through it.	A cusp and $p-2$ distinct points on BC.	A singular point formed by the union of an ordinary multiple point and a binode.	A binode on the section by the plane BCD

6. The first theorem has already been proved, and the others may be established by means of the results of S.C. and M.P. For brevity we shall write $\lambda=\frac{1}{2}p\,(p-1)$, $\mu=\frac{1}{2}p\,(p-1)^2$.

(ii) The first two portions are proved in S.C., § 2, where it is shown that when $2r$ tangents coincide in pairs, each pair being distinct, the constituents of the multiple points are

$$\delta=\lambda-r,\quad \kappa=r.$$

In M.P., § 3, it is shown that when the tangent cone has δ distinct nodal generators, the constituents of the multiple point on the surface are

$$C=\mu-\delta,\quad B=\delta.$$

(iii) From S.C., § 2, and M.P., § 3, it follows that if a multiple point on a curve has r tangents, each of which consists of three coincident tangents, the constituents of the multiple point are

$$\delta=\lambda-2r,\quad \kappa=2r,$$

whilst, if the tangent cone at the multiple point on the surface has κ cuspidal generators, its constituents are

$$C=\mu-2\kappa,\quad B=2\kappa.$$

(iv) The first two portions follow from S.C., § 2; and the constituents of the multiple point on the curve are

$$\delta=\lambda-2r+1,\quad \kappa=2r-1,$$

whilst the latter part follows from M.P., § 4, which shows that the constituents of the multiple point on the surface are

$$C=\mu-2\delta+1,\quad B=2\delta-1.$$

(v) By S.C., § 3, (i), it follows that if a multiple point on a curve has s pairs of tacnodal branches and $p-2s$ ordinary branches, all of which are distinct, its constituents are

$$\delta=\lambda+s,\quad \kappa=0,$$

and it follows from the theorem S.S., § 13, that if the tangent cone at a multiple point on a surface has ε nodal generators, all of which are lines of closest contact, the constituents are

$$C=\mu+\delta,\quad B=0.$$

(vi) Putting $2r$ for r in S.C., § 4, it follows that the constituents of the multiple point on the curve are

$$\delta = \lambda - 2r + 1, \quad \kappa = 2r.$$

Let the tangent cone at the multiple point on the surface have $\delta + 1$ distinct nodal generators, and let one of them AB be a line of closest contact. Then it is shown in M.P., § 5, that if the cone possesses $\delta + 1$ nodal generators, which move up to coincidence along a continuous curve, the total reduction of class is $2\mu + 2\delta + 1$; whence $s = 2\delta + 1$. Also it follows from M.P., § 6, (i), that if one of the generators before coincidence is a line of closest contact the constituents of the singularity are

$$C = \mu - 2\delta + 1, \quad B = 2\delta,$$

since we have shown that $s = 2\delta + 1$.

(vii) The first two portions follow from S.C., § 5, which shows that the constituents of the multiple point on the curve are

$$\delta = \lambda + s - 2r + 2, \quad \kappa = 2r - 2.$$

To prove the third part, it follows from M.P., § 6, (ii), that if the tangent cone at the multiple point has r coincident nodal generators, all of which before coincidence are lines of closest contact,

$$2C + 3B = 2\mu + 4r - 2,$$

$$C + B = \mu + r.$$

But if the tangent cone possesses $\delta - r$ additional distinct nodal generators, all of which are lines of closest contact, their effect is to produce an additional reduction of class equal to $2\delta - 2r$, and to add $\delta - r$ double points to the constituents of the multiple point; whence

$$2C + 3B = 2\mu + 2\delta + 2r - 2,$$

$$C + B = \mu + \delta,$$

accordingly

$$C = \mu + \delta - 2r + 2, \quad B = 2r - 2.$$

(viii) If in S.C., § 4, we put $r = 1$, we obtain the singularity in question, and its point constituents are

$$\delta = \lambda, \quad \kappa = 1,$$

and it follows from M.P., § 6, (i), that if the tangent cone at a multiple point possesses a cuspidal generator, which is a line of closest contact, the constituents are

$$C=\mu, \quad B=1.$$

(ix) The first two portions are proved in S.C., § 9; and the last two portions in § 4, (vi), of the present paper. The point constituents of the singularities are $\delta=\lambda$, $\kappa=1$ for a curve; and $C=\mu$, $B=1$ for a surface.

7. It may, at first sight, appear strange that two such different singularities as the ones discussed in (v) and (vi) of § 4 should have the same point constituents; but the theory of curves supplies the explanation. The singularity corresponding to (v) in plane geometry is a multiple point consisting of one pair of tacnodal branches, one coincident ordinary branch, and $p-3$ distinct ordinary branches; and its constituents are given by the equations

$$\delta=\lambda, \quad \kappa=1, \quad \tau=2, \quad \iota=0,$$

whilst the one corresponding to (vi) is a multiple point consisting of a rhamphoid cusp and $p-2$ distinct ordinary branches passing through it, and its constituents are given by the equation

$$\delta=\lambda, \quad \kappa=1, \quad \tau=1, \quad \iota=1.$$

Both singularities are therefore *mixed* ones, whose point constituents are the same, but whose line constituents are different. The theory of the plane singularities of surfaces awaits investigation, and I am unable at the present time to offer any contribution to the subject; but it is beyond doubt that the singularities (v) and (vi) of § 4 in solid geometry are *mixed* ones, whose constituents consist partly of point and partly of plane singularities; and as soon as the theory of the latter class of singularities is discovered, it is practically certain that the plane constituents of (v) and (vi) will be found to be different.

8. The process of constructing singularities by the addition of conic nodes and binodes may be carried on to any extent. Thus, if the transformed surface has a binode at B, whose axis has quadritactic contact with it, the effect is to add to the multiple point on the original surface two conic nodes, which

move up to coincidence with the multiple point along a continuous curve. And when the multiple point is a conic node we obtain a singularity consisting of three conic nodes, which move up to coincidence along a continuous curve and is analogous to an oscnode in plane geometry. The equation of the surface may be obtained as follows. In (8) put

$$p=2,\quad v_2=\gamma\delta,\quad w_0=0,\quad w_1=W(\gamma+\delta),$$

$$\left.\begin{aligned} w_2&=L\gamma^2+M\gamma\delta+N\delta^2\\ W_1&=(\lambda\gamma+\mu\delta)\,W\end{aligned}\right\}\ldots\ldots\ldots\ldots(19),$$

then (8) becomes

$$\alpha^{n-2}u_n+\alpha^{n-1}\Omega u_{n-1}+\ldots+\alpha^2\Omega^{n-4}(\beta^4W^2+\beta^3W_1+\ldots W_4)$$

$$+\alpha\Omega^{n-3}\{W\beta^2(\gamma+\delta)+\beta w_2+w_3\}+\Omega^{n-2}\gamma\delta=0\ldots(20).$$

Arranging in powers of β, (20) becomes

$$(\alpha W+\gamma)(\alpha W+\delta)\beta^{2n-4}+\{(n-2)(\alpha W+\gamma)(\alpha W+\delta)\Omega_1$$

$$-\alpha W(2\alpha W+\gamma+\delta)\Omega_1+\alpha w_2+\alpha^2W_1\}\beta^{2n-5}+\ldots\ldots=0\ldots(21).$$

The equations of the axis of the binode are

$$-\alpha W=\gamma=\delta,$$

and if these values, together with those of w_2 and W_1 from (19), are substituted in (20), the conditions that the axis of the binode should have quadritactic contact with the surface are that the coefficients of β^{2n-4} and β^{2n-5} should vanish, which requires that

$$L+M+N=\lambda+\mu\ldots\ldots\ldots\ldots\ldots(22).$$

The equation of the original surface now becomes

$$\alpha^{n-4}(\alpha\gamma+W\beta^2)(\alpha\delta+W\beta^2)+\alpha^{n-3}(\beta w_2+w_3)+\alpha^{n-4}(\beta^3W_1+\beta^2W)+\ldots$$

$$+\alpha^{n-5}u_5+\ldots\ldots u_n=0\ldots\ldots\ldots\ldots(23),$$

where the values of w_2 and W_1 are given by (19), subject to the condition (22).

Substituting the value of M from (22), we obtain

$$\alpha w_2+\beta^2W_1=\alpha(L\gamma-N\delta)(\gamma-\delta)+\lambda\gamma(\alpha\delta+W\beta^2)+\mu\delta(\alpha\gamma+W\beta^2),$$

and (23) reduces to the form

$$\alpha^{n-4}(\alpha\gamma + \lambda\beta\gamma + W\beta^2)(\alpha\delta + \mu\beta\gamma + W\beta^2)$$
$$+ \alpha^{n-3}\{\beta(L\gamma - N\delta)(\gamma - \delta) + w_3\}$$
$$+ \alpha^{n-4}(\beta^2 V_2 + \beta V_3 + V_4) + \alpha^{n-5}u_5 + \ldots\ldots u_n = 0 \ldots\ldots (24),$$

where $V_n = (\gamma, \delta)^n$.

The section of the surface by the plane $\gamma = \delta$ is a curve having an oscnode at A; and, when the surface is a quartic, the equation of the section can be reduced to the standard form

$$(\alpha\gamma + u_2)^2 + \gamma^3 u_1 = 0,$$

where $u_n = (\beta, \gamma)^n$.

9. The quadric transformation does not explain why a multiple generator of order q on the tangent cone produces a further reduction of class equal to $(q-1)^2$; but as soon as this has been established by other methods, it will be found that a theory of multiple generators on the tangent cone exists analogous to the theory of multiple points developed in S.C. One example will be sufficient.

Let the equation of the tangent cone at a multiple point of order $q+2$ be

$$\beta^2 v_1^2 w_q + \beta v_1 w_{q+2} + w_{q+4} = 0 \ldots\ldots\ldots\ldots (25),$$

where $v_1 = \gamma + \delta$, and $w_q = (\gamma, \delta)^q$. Proceed as in M.P., § 4, by writing down the first polars of C and D, and eliminating β, and it will be found that the eliminant is

$$A_{2q+1} B_{2q+5} = C^2_{2q+3},$$

where A, B, and C are binary quantics of (γ, δ) of the degrees indicated by the suffixes. This shows that the first polars intersect in $4q+6$ ordinary points, and that B absorbs

$$(q+3)^2 - 4q - 6 = (q+1)^2 + 2 \ldots\ldots\ldots\ldots (26)$$

points; hence the right-hand side of (26) gives the reduction of class produced by the multiple generator on the cone.

In the same way it can be shown that, if $w_q = v_1^r w_{q-r}$, so that r ordinary sheets of the cone coincide with the pair of tacnodal sheets, where $r > 1$, the reduction of class produced by the generator is

$$(q+1)^2 + r + 2.$$

Putting $q-2$ for q, we thus obtain the following theorem :—

If the tangent cone at a multiple point of order p possesses a singular generator, which consists of a multiple line of order q having one pair of tacnodal sheets, r ordinary sheets which coincide with the pair of tacnodal sheets, and $q-r-2$ distinct ordinary sheets, the reduction of class is

$$p(p-1)^2+(q-1)^2+r+2.$$

When the generator is not a line of closest contact, the value of $C+B$ is $\frac{1}{2}p(p-1)^2$; hence C and B are known.

The foregoing result may be easily extended to the cases in which the cone has any multiple generator of the kind considered in S.C.

10. The theory of the quadric transformation also fails when the tangent cone at the multiple point is of the form u_p^2. In (7) and (8) let $p=2$, $u_p=\delta^2$, so that the singularity at A in (7) is a unode, whose constituents, as we already know, are three conic nodes. The last three terms of (8) now become

$$\alpha^2\Omega^{n-4}u_4+\alpha\Omega^{n-3}u_3+\Omega^{n-2}\delta^2,$$

and if B be one of the points when the line BC intersects the cone u_3, it follows that the highest power of β is β^{2n-4}, and that its coefficient is

$$\alpha^2W^2+\alpha w_1+\delta^2.$$

Hence B is a conic node on the section of the transformed surface by the plane BCD, whose nodal cone touches this plane; and, since there are three of such points, it ought to follow from § 4, (v), that the constituents of a unode are $C=1$, $B=3$, which shows that the theory fails.

In a paper which is shortly to be published in the *Rendiconti del Circolo Matematico di Palermo*, I have obtained a proof of the theorem that the point constituents of a multiple point of order p, the tangent cone at which is *anautotomic*, are $\frac{1}{2}p(p-1)^2$ conic nodes. This completes the proofs of the theorems given in M.P. and the present paper.

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NOTE ON LEFT AND RIGHT-HANDED SEMI-CONTINUOUS FUNCTIONS.

By W. H. Young, Sc.D., F.R.S.

§ 1. In a previous paper in the *Quarterly Journal** I showed that the upper (lower) right-handed associated limiting function of a function of a single real variable differed from the upper (lower) left-handed associated limiting function at only a countable set of points, or with the notation of that paper

$$\phi_R = \phi_L = \phi, \quad \psi_R = \psi_L = \psi,$$

except at a countable set of points.†

I showed further that ϕ is upper semi-continuous, and that at each of its points of continuity both ϕ_R and ϕ_L are continuous, as was there pointed out. Hence ϕ_R, ϕ_L, and similarly ψ_R, ψ_L, are point-wise discontinuous functions.

In the following note I give a general theorem, of which this is a special case, viz. I show that *any* right or left-handed semi-continuous function belongs to Baire's first class, that is, it is point-wise discontinuous with respect to every perfect set.

§ 2. *Lemma. If f is lower semi-continuous on the right, and finite, $\phi_R - f$ has the lower bound zero.*

For, if P be any point, it is known‡ that any limit approached by ϕ_R, or by f, as x approaches P as limit on the left is $\leq \phi_R(P)$.

But, by the definition of $\phi_R(P)$, we can so choose the sequence described by x that $f(x)$ has the limit $\phi_R(P)$, hence $\phi_R(x)$, which, when f is lower semi-continuous on the right, is never less than $f(x)$, has the same limit. Since this limit is, by hypothesis, finite, it follows that $\phi_R(x) - f(x)$ as x describes this sequence, sinks below any positive quantity, which proves the lemma.

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* In the present Volume, pp. 67–83.

† I may say that I have succeeded in extending this result still further, having proved that *the two multi-valued functions defined at each point by all the limits on the left and all the limits on the right respectively differ only at a countable set of points.* The statement that at a point of discontinuity *there is no distinction of right and left except at a countable set of points* is now therefore to be interpreted in this extended sense.

‡ *Loc. cit*, p. 71.

Similarly:—

If f is upper semi-continuous on the right and finite, $f-\psi_R$ has the lower bound zero.

Two similar results hold interchanging "left" and "right."

It is to be remarked that, although the associated functions are usually defined with respect to the continuum, they may equally well be taken with reference to any perfect set, as Baire points out in his original memoir; thus the above lemma may be taken to refer to a segment or to any perfect set, and in the theorems which follow the terms "continuous" and "set of the first category" may be taken to mean in reference to the continuum, or to any perfect set.

§ 3. We now state and prove the theorem in question.

THEOREM. *If f is a function which is lower (upper) semi-continuous on the right (left), it is continuous except at a set of the first category.*

Since f is lower semi-continuous on the right,

$$f \leq \psi_R \leq \phi_R$$

everywhere, therefore

$$\phi_R - f \geq 0,$$

and determinate in any interval throughout which ϕ_R is finite. In such an interval, moreover, $\phi_R - f$ is upper semi-continuous on the right, so that the points

$$\phi_R - f \geq k$$

form a set which is closed on the left.* If this set were dense everywhere in any part of the interval considered, it would therefore fill up the partial interval, which is impossible, by the preceding lemma.

Thus the points

$$\phi_R - f \geq k$$

are for each value of k dense nowhere, so that the points at which

$$\phi_R - f > 0,$$

form a set of the first category. At the remaining points, since ϕ_R cannot be less than ψ_R,

$$\phi_R = f = \psi_R,$$

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* *Loc. cit.*, p. 72.

so that f is continuous on the right. Since there is no distinction of right and left except at a countable set of points it follows that f is continuous except at a set of the first category. Thus the theorem holds in any interval throughout which ϕ_R is finite, and therefore in any interval in which the points $\phi_R = \pm\infty$ are dense nowhere.

Now at any point where $\phi_R = -\infty$, $f = -\infty$, and is therefore continuous on the right. But the points at which the lower semi-continuous function $f = -\infty$ form a set which is closed on the left, and therefore fills up any interval in which it is dense everywhere. Throughout such an interval $\phi_R = -\infty$, so that f is continuous. Thus the theorem is true in any interval in which the points $\phi_R = -\infty$ are not dense nowhere.

Similarly it is true in any interval in which the points $\phi_R = +\infty$ are not dense nowhere. For the points $\phi_R = +\infty$ form a set which is closed on the left, and therefore fills up an interval in which it is dense nowhere. In such an interval the points $f \leq k$, which, f being lower semi-continuous on the right, form a set closed on the left, must be dense nowhere. For otherwise at every point of some partial interval we should have $f \leq k$, and therefore $\phi_R \leq k$. Thus in such an interval the points $f \neq +\infty$ form a set of the first category, in accordance with the theorem.

Since, by what has been shown, the whole continuum may be divided up into a set of intervals in which $\phi_R = +\infty$, a set in which $\phi_R = -\infty$, and a set in which the points $\phi_R = \pm\infty$ are dense nowhere, and the theorem holds in each interval separately, it holds in the whole continuum.*

Cor. A function which is upper (lower) semi-continuous on one side, and whose lower (upper) bound in every interval is zero, is a null function.

Göttingen.

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* It should be noted that the theorem just proved is not a consequence of the main result of the paper quoted. A function may be totally discontinuous and yet upper (lower) semi-continuous except at a countable set of points :—*e.g.* the familiar function, which is unity at the irrational and zero at the rational points.

ON ELLIPTIC-FUNCTION EXPANSIONS IN WHICH THE COEFFICIENTS ARE POWERS OF THE COMPLEX NUMBERS HAVING n AS NORM.

By J. W. L. Glaisher.

Introduction, § 1.

§ 1. ON p. 33 of the last volume (Vol. XXXVIII.) I have assumed the truth of the formula

$$k^2k'^2\rho^5 = 16\Sigma_1^\infty \chi_4(n)\, q^n,$$

where $\chi_4(n) = \frac{1}{4}\Sigma t^4$, t being any complex number having n as norm, and stated that the proof would be given in a separate paper which would be devoted to the consideration of the class of elliptic-function expansions in which the coefficients are powers of the complex numbers having n as norm.

In the same volume (p. 300) I have also assumed the formula

$$(4k^2k'^2 + k^4k'^4)\,\rho^9 = 64\Sigma_1^\infty \chi_8(n)\, q^n,$$

where $\chi_8(n) = \frac{1}{4}\Sigma t^8$, leaving the proof to the same subsequent paper.

§ 2. The present paper is the one referred to. Its principal object is to investigate the q-expansions in elliptic functions which have as coefficients $\chi_r(n)$ where $\chi_r(n) = \frac{1}{4}\Sigma t^r$.

The proofs of the two formulæ which have been assumed in the previous papers are given in §§ 13–14.

The second portion of the paper (§§ 28–34) relates to the functions $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$, of which considerable use has been made in previous papers of this series. The definitions of the functions as coefficients are derived directly from their definitions in terms of representations by any four squares.

The third portion (§§ 35–49) relates to the function $\Theta(n)$, which was introduced for the purpose of expressing the numbers of representations by sixteen squares. From the coefficient-definition of $\Theta(n)$ arithmetical definitions are obtained by which it is expressed in terms of representations by any four squares, and, when n is uneven, also by four uneven squares.

In the concluding portion (§§ 50–69) the functions $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$ are expressed as sums of the first, second,

third, and fourth powers of certain complex quantities derived from representations by four squares.

Expansions in which the coefficients are powers of the complex numbers having the exponent as norm, §§ 3–26.

Notation, &c., §§ 3–4.

§ 3. The notation used is as follows, K and E being as defined in Jacobi's *Fundamenta Nova*.

$$\rho=\frac{2K}{\pi},\quad G=E-k'^2K,\quad I=E-K,$$

$$\frac{4KE}{\pi^2}=e,\quad \frac{4KG}{\pi^2}=g,\quad \frac{4KI}{\pi^2}=i,$$

so that $$e=g+k'^2\rho^2,\quad i=g-k^2\rho^2.$$

§ 4. The expansion formulæ upon which the results obtained in the paper depend will be deduced from the following general theorem in which ζ denotes $q\dfrac{d}{dq}$:

$$\zeta(k^mk'^nr^ri^\alpha g^\beta e^\gamma)=\tfrac{1}{2}k^mk'^nr^r\{(\beta+\gamma+n)\,i^{\alpha+1}g^\beta e^\gamma$$
$$+(\gamma+\alpha-n-m+r)\,i^\alpha g^{\beta+1}e^\gamma+(\alpha+\beta+m)\,i^\alpha g^\beta e^{\gamma+1}$$
$$-\alpha i^{\alpha-1}g^{\beta+1}e^{\gamma+1}-\beta i^{\alpha+1}g^{\beta-1}e^{\gamma+1}-\gamma i^{\alpha+1}g^{\beta+1}e^{\gamma-1}\}.*$$

Values of $\Sigma_{-\infty}^{\infty}n^2q^{n^2}$, $\Sigma_{-\infty}^{\infty}n^4q^{n^2}$, *&c.*, §§ 5–9.

§ 5. I start with the four fundamental formulæ

$$\text{(i)}\quad \Sigma_{-\infty}^{\infty}q^{n^2}=\rho^{\frac{1}{2}},$$
$$\text{(ii)}\quad \Sigma_{-\infty}^{\infty}(-1)^nq^{n^2}=k'^{\frac{1}{2}}\rho^{\frac{1}{2}},$$
$$\text{(iii)}\quad \Sigma_{-\infty}^{\infty}q^{\frac{1}{4}m^2}=k^{\frac{1}{4}}\rho^{\frac{1}{2}},$$
$$\text{(iv)}\quad \Sigma_{-\infty}^{\infty}m'q^{\frac{1}{4}m^2}=k^{\frac{1}{4}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}},$$

where (as also throughout the rest of the paper) n denotes any number, m any uneven number and $m'=(-1)^{\frac{1}{2}(m-1)}m$, so that $m'=m$ when m is of the form $4k+1$ and $=-m$ when m is of the form $4k+3$.

In all the q-series the sign Σ refers to n or m.

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* This formula is given in this form on p. 249 of my book on the Zeta Function (still unpublished). It may be easily deduced from the formula on p. 353 (§ 44) of Vol. xx. of the *Quarterly Journal*.

§ 6. Operating with ζ upon (1) we obtain $\Sigma_{-\infty}^{\infty} n^2 q^{n^2}$, operating again, $\Sigma_{-\infty}^{\infty} n^4 q^{n^2}$, and so on. Treating similarly (ii), (iii), (iv) we may obtain by repeated use of the theorem in § 4 the values of

$$\Sigma_{-\infty}^{\infty}(2n)^{2r}q^{\frac{1}{4}(2n)^2}, \quad \Sigma_{-\infty}^{\infty}(-1)^n(2n)^{2r}q^{\frac{1}{4}(2n)^2}, \quad \Sigma_{-\infty}^{\infty}m^{2r}q^{\frac{1}{4}m^2},$$
$$\Sigma_{-\infty}^{\infty}m^{2r+1}q^{\frac{1}{4}m^2},$$

for $r=1, 2, 3, \ldots$.

§ 7. Operating with ζ on (i), ..., (iv), we find

$$\begin{aligned}
\zeta(\rho^{\frac{1}{2}}) &= \tfrac{1}{4}\rho^{\frac{1}{2}}g,\\
\zeta(k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{4}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}i,\\
\zeta(k^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{4}k^{\frac{1}{2}}\rho^{\frac{1}{2}}e,\\
\zeta(k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{4}k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}(i+g+e),
\end{aligned}$$

and, operating again,

$$\begin{aligned}
\zeta^2(\rho^{\frac{1}{2}}) &= \tfrac{1}{16}\rho^{\frac{1}{2}}(g^2+2ge-2ei+2ig),\\
\zeta^2(k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{16}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}(i^2-2ge+2ei+2ig),\\
\zeta^2(k^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{16}k^{\frac{1}{2}}\rho^{\frac{1}{2}}(e^2+2ge+2ei-2ig),\\
\zeta^2(k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{16}k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}(i^2+g^2+e^2+4ge+4ei+4ig),
\end{aligned}$$

and, again,

$$\begin{aligned}
\zeta^3(\rho^{\frac{1}{2}}) &= \tfrac{1}{64}\rho^{\frac{1}{2}}(g^3+2ige+6g^2e+6g^2i+8e^2g+8i^2g-8ie^2-8i^2e),\\
\zeta^3(k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{64}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}(i^3+2ige+6i^2e+6i^2g+8e^2i+8g^2i-8ge^2-8g^2e),\\
\zeta^3(k^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{64}k^{\frac{1}{2}}\rho^{\frac{1}{2}}(e^3+2ige+6ge^2+6ie^2+8g^2e+8i^2e-8gi^2-8g^2i),\\
\zeta^3(k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{64}k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}(i^3+g^3+e^3+9i^2g+9ig^2+9g^2e+9ge^2+9i^2e+9ie^2\\
&\qquad +48ige),
\end{aligned}$$

and so on.

§ 8. It is, however, rather more convenient to express each result by means of one only of the three quantities i, g, e. Thus we have

$$\begin{aligned}
\zeta^2(\rho^{\frac{1}{2}}) &= \tfrac{1}{16}\rho^{\frac{1}{2}}(3g^2+2k^2k'^2\rho^4),\\
\zeta^2(k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{16}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}(3i^2-2k^2\rho^4),\\
\zeta^2(k^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{16}k^{\frac{1}{2}}\rho^{\frac{1}{2}}(3e^2-2k'^2\rho^4),\\
\zeta^2(k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}) &= \tfrac{1}{16}k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{1}{2}}\{15g^2+10g(k'^2-k^2)\rho^2+(1-6k^2k'^2)\rho^4\},
\end{aligned}$$

&c. &c.

§ 9. By operating upon (i), . ., (iv) with ζ, ζ^2, ... and expressing the results in terms of only one of the letters i, g, e, we obtain the following results:—

$$\Sigma_{-\infty}^{\infty} q^{n^2} = \rho^{\frac{1}{4}},$$

$$2^2 \Sigma_{-\infty}^{\infty} n^2 q^{n^2} = \rho^{\frac{1}{4}} g,$$

$$2^4 \Sigma_{-\infty}^{\infty} n^4 q^{n^2} = \rho^{\frac{1}{4}} (3g^2 + 2k^2 k'^2 \rho^4),$$

$$2^6 \Sigma_{-\infty}^{\infty} n^6 q^{n^2} = \rho^{\frac{1}{4}} \{15g^3 + 30gk^2 k'^2 \rho^4 + 8k^2 k'^2 (k'^2 - k^2)\, \rho^6\},$$

$$2^8 \Sigma_{-\infty}^{\infty} n^8 q^{n^2} = \rho^{\frac{1}{4}} \{105g^4 + 420g^2 k^2 k'^2 \rho^4 + 224gk^2 k'^2 (k'^2 - k^2)\, \rho^6 + (32k^2 k'^2 - 132k^4 k'^4)\, \rho^8\},$$

&c. &c.

$$\Sigma_{-\infty}^{\infty} (-1)^n q^{n^2} = k'^{\frac{1}{4}} \rho^{\frac{1}{4}},$$

$$2^2 \Sigma_{-\infty}^{\infty} (-1)^n n^2 q^{n^2} = k'^{\frac{1}{4}} \rho^{\frac{1}{4}} i,$$

$$2^4 \Sigma_{-\infty}^{\infty} (-1)^n n^4 q^{n^2} = k'^{\frac{1}{4}} \rho^{\frac{1}{4}} (3i^2 - 2k^2 \rho^4),$$

$$2^6 \Sigma_{-\infty}^{\infty} (-1)^n n^6 q^{n^2} = k'^{\frac{1}{4}} \rho^{\frac{1}{4}} \{15i^3 - 30ik^2 \rho^4 - 8k^2 (1 + k^2)\, \rho^6\},$$

$$2^8 \Sigma_{-\infty}^{\infty} (-1)^n n^8 q^{n^2} = k'^{\frac{1}{4}} \rho^{\frac{1}{4}} \{105i^4 - 420i^2 k^2 \rho^4 - 224ik^2 (1 + k^2)\, \rho^6 - (32k^2 k'^4 + 132k^4)\, \rho^8\},$$ *

&c. &c.

$$\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} \rho^{\frac{1}{4}},$$

$$\Sigma_{-\infty}^{\infty} m^2 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} \rho^{\frac{1}{4}} e,$$

$$\Sigma_{-\infty}^{\infty} m^4 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} \rho^{\frac{1}{4}} (3e^2 - 2k'^2 \rho^4),$$

$$\Sigma_{-\infty}^{\infty} m^6 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} \rho^{\frac{1}{4}} \{15e^3 - 30ek'^2 \rho^4 + 8k'^2 (1 + k'^2)\, \rho^6\},$$

$$\Sigma_{-\infty}^{\infty} m^8 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} \rho^{\frac{1}{4}} \{105e^4 - 420e^2 k'^2 \rho^4 + 224ek'^2 (1 + k'^2)\, \rho^6 - (32k'^2 k^4 + 132k'^4)\, \rho^8,$$

&c. &c.

$$\Sigma_{-\infty}^{\infty} m' q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} k'^{\frac{1}{4}} \rho^{\frac{3}{4}},$$

$$\Sigma_{-\infty}^{\infty} m'^3 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} k'^{\frac{1}{4}} \rho^{\frac{3}{4}} \{3g + (k'^2 - k^2)\, \rho^2\},$$

$$\Sigma_{-\infty}^{\infty} m'^5 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} k'^{\frac{1}{4}} \rho^{\frac{3}{4}} \{15g^2 + 10g\,(k'^2 - k^2)\, \rho^2 + (1 - 6k^2 k'^2)\, \rho^4\},$$

$$\Sigma_{-\infty}^{\infty} m'^7 q^{\frac{1}{4}m^2} = k^{\frac{1}{2}} k'^{\frac{1}{4}} \rho^{\frac{3}{4}} \{105g^3 + 105g^2 (k'^2 - k^2)\, \rho^2 + 21g\,(1 - 6k^2 k'^2)\, \rho^4 + (k'^2 - k^2)\,(1 - 10k^2 k'^2)\, \rho^6\},$$

&c. &c.

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* The second set of formulæ may be derived from the first by changing the sign of q. By the change of q into $-q$, e is unaltered, but i and g are interchanged, and ρ, $k'\rho$, $k\rho$ become $k'\rho$, ρ, $\sqrt{(-1)}\,k\rho$ respectively.

Complex numbers, § 10.

§ 10. If $n=a^2+b^2$, where a and b are different and neither zero, there are, corresponding to this partition of n, eight complex numbers which have n as norm, viz. the four associated numbers

$$a+ib,\ ia-b,\ -a-ib,\ -ia+b,$$

and their four conjugates. If $a=b$ (in which case n is the double of a square), or if one is zero (in which case n is a square) there are, corresponding to this partition, only four complex numbers having n as norm.

The sum of the eight (or four, if $a=b$ or if one is zero) complex numbers is zero, as also is the sum of their squares. The same is true also whenever the exponent is even, or when it is even and of the form $4k+2$, but not if it is of the form $4k$.

Expressions for $\chi_4(n)$ *and* $\chi_8(n)$, §§ 11–12.

§ 11. If a and b are different and non-zero, then the sum of the fourth powers of the eight complex numbers, corresponding to the partition $n=a^2+b^2$, is

$$8\,(a^4-6a^2b^2+b^4),$$

and the sum of the eighth powers of these numbers is

$$8\,(a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8).$$

If $a=b$, or if one is zero, these quantities are to be halved.

§ 12. If then we define $\chi_{4k}(n)$ as $\frac{1}{4}\Sigma t^{4k}$, where t is any complex number having n as norm, we have

$$\chi_4(n)=2\Sigma\,(a^4-6a^2b^2+b^4),$$

and

$$\chi_8(n)=2\Sigma\,(a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8),$$

where Σ applies to all the partitions of n, it being understood that a term corresponding to a partition in which the two parts are equal or one is zero is to be halved.

Proofs of the formulæ quoted in § 1, §§ 13–14.

§ 13. The value of $\chi_4(n)$ which has just been obtained shows that $4\chi_4(n)$ is equal to the coefficient of q^n in the

expression in ascending powers of q of

$$2\Sigma_{-\infty}^{\infty} n^4 q^{n^2} \times \Sigma_{-\infty}^{\infty} q^{n^2} - 6\,(\Sigma_{-\infty}^{\infty} n^2 q^{n^2})^2,$$

that is, substituting for the series, their values given in § 9,

$$4\Sigma_1^{\infty} \chi_4(n)\, q^n = \tfrac{1}{16}\rho\,(6g^2 + 4k^2k'^2\rho^4 - 6g^2)$$
$$= \tfrac{1}{4}k^2k'^2\rho^5,$$

that is, $$k^2k'^2\rho^5 = 16\Sigma_1^{\infty} \chi_4(n)\, q^n,$$

which is the first of the formulæ quoted in § 1.

§ 14. Similarly $4\chi_8(n)$ is equal to the coefficient of q^n in the expansion of

$$2\Sigma_{-\infty}^{\infty} n^8 q^{n^2} \times \Sigma_{-\infty}^{\infty} q^{n^2} - 56\Sigma_{-\infty}^{\infty} n^6 q^{n^2} \times \Sigma_{-\infty}^{\infty} n^2 q^{n^2} + 70\,(\Sigma_{-\infty}^{\infty} n^4 q^{n^2})^2,$$

whence, substituting from § 9 and reducing, we find

$$4\Sigma_1^{\infty} \chi_8(n)\, q^n = \tfrac{1}{256}\rho\,(64k^2k'^2 + 16k^4k'^4)\,\rho^8,$$

that is, $$(4k^2k'^2 + k^4k'^4)\,\rho^9 = 64\Sigma_1^{\infty} \chi_8(n)\, q^n,$$

which is the second of the formulæ quoted in § 1.

The function $\chi_0(n)$, § 15.

§ 15. The function $\chi_0(n) = \frac{1}{4}\Sigma t^0$ and is therefore equal to one fourth of the number of complex numbers which have n as norm. Thus

$$1 + 4\Sigma_1^{\infty} \chi_0(n)\, q^n = (\Sigma_{-\infty}^{\infty} q^{n^2})^2 = \rho.$$

The function $\chi_0(n)$ is therefore equal to $E_0(n)$ where $E_0(n)$ denotes the excess of the number of real divisors of n of the form $4k+1$ over the number of real divisors of the form $4k+3$.

Primary complex numbers, § 16.

§ 16. If m is an uneven number and $a^2 + b^2$ is a partition of m, a being uneven and b even, then among the eight complex numbers connected with this partition the two

$$(-1)^{\frac{1}{4}(a+b-1)}(a + ib) \text{ and } (-1)^{\frac{1}{4}(a+b-1)}(a - ib)$$

were termed primary by Gauss. If b is zero (in which case m is a square), there is but one primary number having m as norm, viz. $(-1)^{\frac{1}{4}(a-1)}a$.

Definition of $\chi_r(m)$, § 17.

§ 17. I define $\chi_r(m)$ as the sum of the r^{th} powers of the primary complex numbers having m as norm. This definition is in accord with that of $\chi_{4k}(n)$ in § 12, for (including the case of $k=0$) the sum of the $(4k)^{\text{th}}$ powers of all the complex numbers having m as norm is equal to four times the sum of the primary complex numbers having the same norm. Thus, if a^2+b^2 is a partition of m in which a is uneven and b even, we have

$$\chi_1(m)=2\Sigma(-1)^{\frac{1}{2}(a+b-1)}a,$$

$$\chi_2(m)=2\Sigma(a^2-b^2),$$

$$\chi_3(m)=2\Sigma(-1)^{\frac{1}{2}(a+b-1)}(a^3-3ab^2),$$

$$\chi_4(m)=2\Sigma(a^4-6a^2b^2+b^4),$$

$$\chi_5(m)=2\Sigma(-1)^{\frac{1}{2}(a+b-1)}(a^5-10a^3b^2+5ab^4),$$

$$\chi_6(m)=2\Sigma(a^6-15a^4b^2+15a^2b^4-b^6),$$

$$\chi_7(m)=2\Sigma(-1)^{\frac{1}{2}(a+b-1)}(a^7-21a^5b^2+35a^3b^4-7ab^6),$$

$$\chi_8(m)=2\Sigma(a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8),$$

&c. &c.

The uneven number m must be of the form $4k+1$, for a number of the form $4k+3$ cannot be the sum of two squares.

As already mentioned, when the suffix r is of the form $4k$, the value of $\chi_r(m)$ is included in that of $\chi_r(n)$.

Values of $\chi_r(4n+1)\,q^{\frac{1}{4}(4n+1)}$, §§ 18—19.

§ 18. Taking the quantity $\chi_1(m)$, the first formula of the preceding section shows that

$$2\Sigma_1^\infty\chi_1(4n+1)\,q^{\frac{1}{4}(4n+1)}=\Sigma_{-\infty}^{\infty}m'q^{\frac{1}{4}m^2}\times\Sigma_{-\infty}^{\infty}(-1)^n q^{n^2}$$

$$=k^{\frac{1}{2}}k'^{\frac{1}{4}}\rho^{\frac{1}{2}}.k'^{\frac{1}{4}}\rho^{\frac{1}{2}}=k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^1.$$

The second formula shows that

$$2\Sigma_1^\infty\chi_2(4n+1)q^{\frac{1}{4}(4n+1)}=\Sigma_{-\infty}^{\infty}m^2q^{\frac{1}{4}m^2}\times\Sigma_{-\infty}^{\infty}q^{n^2}-\Sigma_{-\infty}^{\infty}(2n)^2q^{n^2}\times\Sigma_{-\infty}^{\infty}q^{\frac{1}{4}m^2}$$

$$=k^{\frac{1}{2}}\rho^{\frac{1}{2}}e.\rho^{\frac{1}{2}}-\rho^{\frac{1}{2}}g.k^{\frac{1}{2}}\rho^{\frac{1}{2}}=k^{\frac{1}{2}}\rho\,(e-g)=k^{\frac{1}{2}}k'^2\rho^3.$$

Similarly

$$2\Sigma_1^\infty \chi_3(4n+1)\, q^{\frac{1}{4}(4n+1)}$$
$$= \Sigma_{-\infty}^{\infty} m'^3 q^{\frac{1}{4}m^2} \times \Sigma_{-\infty}^{\infty}(-1)^n q^{n^2} - 3\Sigma_{-\infty}^{\infty} m' q^{\frac{1}{4}m^2} \times \Sigma_{-\infty}^{\infty}(2n)^2(-1)^n q^{n^2}$$
$$= k^{\frac{1}{2}}k'(1+k^2)\rho^4,$$

and so on.

We thus obtain the system of expansions:

$$k^{\frac{1}{2}}\rho = 2\Sigma_0^\infty \chi_0(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'\rho^2 = 2\Sigma_0^\infty \chi_1(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^2\rho^3 = 2\Sigma_0^\infty \chi_2(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'(1+k^2)\rho^4 = 2\Sigma_0^\infty \chi_3(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^4\rho^5 = 2\Sigma_0^\infty \chi_4(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'(1-6k^2+k^4)\rho^6 = 2\Sigma_0^\infty \chi_5(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^2(1+14k^2+k^4)\rho^7 = 2\Sigma_0^\infty \chi_6(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'(1+k^2)(1-6k^2+k^4)\rho^8 = 2\Sigma_0^\infty \chi_7(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^4(1-66k^2+k^4)\rho^9 = 2\Sigma_0^\infty \chi_8(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$\&c. \qquad\qquad \&c.$$

The fifth and last of these formulæ may also be derived from the expansions involving $\chi_4(n)$ and $\chi_8(n)$ in §§ 13 and 14.

§ 19. Changing the sign of q in these expansions, we find

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho = 2\Sigma_0^\infty (-1)^n \chi_0(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^2 = 2\Sigma_0^\infty (-1)^n \chi_1(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^3 = 2\Sigma_0^\infty (-1)^n \chi_2(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'^2-k^2)\rho^4 = 2\Sigma_0^\infty (-1)^n \chi_3(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^5 = 2\Sigma_0^\infty (-1)^n \chi_4(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}(1+4k^2k'^2)\rho^6 = 2\Sigma_0^\infty (-1)^n \chi_5(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}(1-16k^2k'^2)\rho^7 = 2\Sigma_0^\infty (-1)^n \chi_6(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}(k'^2-k^2)(1+4k^2k'^2)\rho^8 = 2\Sigma_0^\infty (-1)^n \chi_7(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{\frac{1}{2}}(1+64k^2k'^2)\rho^9 = 2\Sigma_0^\infty (-1)^n \chi_8(4n+1)\, q^{\frac{1}{4}(4n+1)},$$
$$\&c. \qquad\qquad \&c.;$$

and by changing q into q^2 the expansions of § 18 give

$$\begin{aligned}
k\rho &= 4\Sigma_0^\infty \chi_0(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'^{\frac{1}{2}}\rho^2 &= 4\Sigma_0^\infty \chi_1(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'\rho^3 &= 4\Sigma_0^\infty \chi_2(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'^{\frac{1}{2}}(1+k'^2)\,\rho^4 &= 8\Sigma_0^\infty \chi_3(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'^2\rho^5 &= 4\Sigma_0^\infty \chi_4(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'^{\frac{1}{2}}(1-6k'^2+k'^4)\,\rho^6 &= -16\Sigma_0^\infty \chi_5(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'(1-k^2k'^2)\,\rho^7 &= 4\Sigma_0^\infty \chi_6(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'^{\frac{1}{2}}(1+k'^2)(1-6k'^2+k'^4)\,\rho^8 &= -32\Sigma_0^\infty \chi_7(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
kk'^2(1-k^2-4k^4)\,\rho^9 &= 4\Sigma_0^\infty \chi_8(4n+1)\, q^{\frac{1}{4}(4n+1)},\\
\&c. \qquad & \qquad \&c.
\end{aligned}$$

Representation by fourteen squares, § 20.

§ 20. Besides the formulæ involving χ_0 (or E_0), χ_2, χ_4, and χ_8 (which have been used in previous papers for the expression of the numbers of representations by sums of squares), only one of these expansions supplies a new formula relating to such representations. This is the seventh equation of § 18 which is equivalent to

$$(k^{\frac{1}{2}}+13k'^{\frac{5}{2}}-13k'^{\frac{3}{2}}-k'^{\frac{1}{2}})\,\rho^7 = 2\Sigma_0^\infty \chi_6(4n+1)\, q^{\frac{1}{4}(4n+1)},$$

and therefore affords another equation connecting $k^{\frac{1}{2}}\rho^7$, $k^{\frac{5}{2}}\rho^7$, $k'^{\frac{3}{2}}\rho^7$, $k'^{\frac{1}{2}}\rho^7$, in addition to the two (xv) and (xvi) on p. 183 of vol. XXXVIII. One more equation is needed, however, in order to obtain the expansions of these four quantities.

The function $\chi_r(n)$, §§ 21–22.

§ 21. The functions $\chi_4(n)$ and $\chi_8(n)$ were first used in the last volume of the *Quarterly Journal* (vol XXXVIII., pp. 20 and 290). The functions $\chi_0(n) = E_0(n)$, $\chi_1(n)$ under the notation $\chi(n)$, and $\chi_2(n)$ under the notation $\lambda(n)$ have been used in several papers published during the last twenty-three years.

I first used the function $\chi(n)$ in vol. XX. of the *Quarterly Journal* (p. 81), and it formed the subject of a special paper

(pp. 97–167) in that volume. In the course of that paper the function $\lambda(m)$ was introduced (p. 145). In the same paper also (p. 109) the function $E_0(n)$, there denoted by $E(n)$, which had been defined by means of the real divisors of n, was considered in connection with the complex divisors. The χ-notation is referred to in the note on p. 33 of vol. XXXVIII.

§ 22. Although, as was pointed out in § 20, only the functions χ_0, χ_2, χ_4, χ_6, χ_8 are of use in connection with representations by sums of squares, it seemed to be of interest to determine the elliptic function quantities which have $\chi_r(m)$ as coefficients also for uneven values of r; for, next to functions depending upon real divisors, the functions $\chi_r(n)$ and $\chi_r(m)$ are among the simplest arithmetical functions depending upon divisors of n and m which can be constructed.

Another method of obtaining the values of
$\Sigma_1^\infty \chi_4(n)\, q^n$ *and* $\Sigma_1^\infty \chi_8(n)\, q^n$, §§ 23–25.

§ 23. It may be noticed that the values of $\Sigma_1^\infty \chi_4(n)\, q^n$ and $\Sigma_1^\infty \chi_8(n)\, q^n$ can be obtained in a slightly different manner as follows:—

We have (§ 12)

$$\begin{aligned}\chi_4(n) &= 2\Sigma\,(a^4 + b^4 - 6a^2b^2)\\ &= 2\Sigma\,\{(a^2+b^2)^2 - 8a^2b^2\}\\ &= 2\Sigma n^2 - 16a^2b^2\\ &= n^2 E_0(n) - 16\Sigma a^2b^2,\end{aligned}$$

and similarly

$$\chi_8(n) = n^4 E(n) - 64\Sigma a^6b^2 - 64\Sigma a^2b^6 + 128\Sigma a^4b^4,$$

whence it follows that

$$\Sigma_1^\infty \chi_4(n)\, q^n = \Sigma_1^\infty n^2 E_0(n)\, q^n - 2\,(\Sigma_{-\infty}^{\infty} n^2 q^{n^2})^2,$$

$$\Sigma_1^\infty \chi_8(n)\, q^n = \Sigma_1^\infty n^4 E_0(n)\, q^n - 16\Sigma_{-\infty}^{\infty} n^6 q^{n^2} \times \Sigma_{-\infty}^{\infty} n^2 q^{n^2} + 16\,(\Sigma_{-\infty}^{\infty} n^4 q^{n^2})^2.$$

§ 24. In order to obtain the values of the two series involving $E_0(n)$, we have only to apply repeatedly the operator ζ to the formula $\rho = 1 + 4\Sigma_1^\infty E_0(n)\, q^n$.

We thus find

$$1+4\Sigma_1^\infty E_0(n)\,q^n = \rho_7$$
$$4\Sigma_1^\infty n\,E_0(n)\,q^n = \tfrac{1}{2}\rho g_7$$
$$4\Sigma_1^\infty n^2 E_6(n)\,q^n = \tfrac{1}{4}\rho\,(2g^7+k^2k'^2\rho^4),$$
$$4\Sigma_1^\infty n^3 E_0(n)\,q^n = \tfrac{1}{8}\rho\,\{6g^3+9gk^2k'^2\rho^4+2k^2k'^2(k'^2-k^2)\,\rho^6\},$$
$$4\Sigma_1^\infty n^4 E_0(n)\,q^n = \tfrac{1}{16}\rho\,\{24g^4+72g^2k^2k'^2\rho^4+32gk^2k'^2(k'^2-k^2)\,\rho^6 + (4k^2k'^2-15k^4k'^4)\,\rho^8\}_{7^7}$$

&c. &c.

§ 25. Substituting the values of the q-series from the preceding section and from § 9, we find

$$\Sigma_1^\infty \chi_1(n)\,q^n = \tfrac{1}{16}\rho\,(2g^7+k^2k'^2\rho^4-2g^7) = \tfrac{1}{16}k^2k'^2\rho^5,$$

and, after reduction,

$$\Sigma_1^\infty \chi_3(n)\,q^n = \tfrac{1}{64}\,(4k^2k'^2+k^4k'^4)\,\rho^9,$$

which agree with the results obtained in §§ 13 and 14.

Values of $\chi_2(m)$ *and* $\chi_6(m)$ *expressed by partitions of* $2m$, § 26.

§ 26. The values of $\chi_2(4n+1)$ and $\chi_6(4n+1)$ can be expressed by means of the partitions of $8n+2$ as a sum of two unequal squares; for, from the fourth group of formulæ in § 9, we find

$$kk'\rho^3 = (\Sigma_{-\infty}^\infty m' q^{\frac{1}{4}m^2})^2,$$

$$2kk'(1-k^2k'^2)\,\rho^7 = 5\,(\Sigma_{-\infty}^\infty m'^3 q^{\frac{1}{4}m^2})^2 - 3\,(\Sigma_{-\infty}^\infty m'^5 q^{\frac{1}{4}m^2})(\Sigma_{-\infty}^\infty m' q^{\frac{1}{4}m^2}).$$

These equations show that the coefficient of $q^{\frac{1}{2}(4n+1)}$ in the expansion of $kk'\rho^3$ is equal to $8\Sigma\alpha'\beta'$, and in the expansion of $kk'(1-k^2k'^2)\,\rho^7$ is equal to $2\Sigma\{10\alpha'^3\beta'^3-3\,(\alpha'^5\beta'+\alpha'\beta'^5)\}$, where Σ applies to every partition of $8n+2$ into two uneven squares $\alpha^2+\beta^2$. When $4n+1$ is a square there is a partition $\alpha^2+\alpha^2$ and the term derived from this partition is to be halved.

Comparing these coefficients with those in the second system of expansions in § 19, we see that

$$\chi_2(4n+1) = 2\Sigma\alpha'\beta',$$
$$\chi_6(4n+1) = \tfrac{1}{2}\Sigma\{10\alpha'^3\beta'^3-3\,(\alpha'^5\beta'+\alpha'\beta'^5)\}.$$

The first of these results has been obtained or referred to in several previous papers, as *e.g.* vol. XXXVI., pp. 342, 345. It was derived from the value $2\Sigma(a^2-b^2)$ of $\chi_2(m)$ in vol. XX, p. 147, by passing from the partition a^2+b^2 of m to the partition $(a+b)^2+(a-b)^2$ of $2m$. The second result may be derived in the same manner from the value $2\Sigma(a^6-15a^4b^2+15a^2b^4-b^6)$ of $\chi_6(m)$.

The Functions $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$, §§ 27–34.

Arithmetical definitions of the functions $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$, § 27.

§ 27. These four functions have been defined on p. 319 of vol. XXXVI., by the equations

$$k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^3 = 16\Sigma_1^\infty \lambda(m)\, q^m,^*$$

$$k^2k'\rho^4 = 16\Sigma_1^\infty P(m)\, q^m,$$

$$k^2k'^{\frac{3}{2}}\rho^5 = 16\Sigma_1^\infty Q(m)\, q^m,$$

$$k^2k'^2\rho^6 = 16\Sigma_1^\infty \Omega(m)\, q^m,$$

and from these definitions it is deduced on p. 323 that if $\alpha^2+\beta^2+\gamma^2+\delta^2$ is any partition of $4m$ into four uneven squares, and, if $\alpha' = (-1)^{\frac{1}{2}(\alpha-1)}\alpha$, $\beta' = (-1)^{\frac{1}{2}(\beta-1)}\beta$, ..., then

$$\lambda(m) = \tfrac{1}{4}\Sigma^{(c)}[\alpha'],$$

$$P(m) = \tfrac{1}{6}\Sigma^{(c)}[\alpha'\beta'],$$

$$Q(m) = \tfrac{1}{4}\Sigma^{(c)}[\alpha'\beta'\gamma'],$$

$$\Omega(m) = \Sigma^{(c)}\alpha'\beta'\gamma'\delta',$$

where
$$[\alpha'] = \alpha'+\beta+\gamma'+\delta,$$

$$[\alpha'\beta'] = \alpha'\beta'+\alpha'\gamma'+\alpha'\delta'+\beta'\gamma'+\beta'\delta'+\gamma'\delta',$$

and
$$[\alpha'\beta'\gamma'] = \alpha'\beta'\gamma'+\alpha'\beta'\delta'+\alpha'\gamma'\delta'+\beta'\gamma'\delta',$$

and the summation symbol $\Sigma^{(c)}$ refers to all the compositions of $4m$ as a sum of four uneven squares.

By algebraical transformations it is shown (pp. 329–334) that $\lambda(m)$, ... may be expressed by means of the representa-

* The function $\lambda(m)$ is the same as $\chi_2(m)$ (§ 21). In the rest of this paper I retain the notation $\lambda(m)$ for the sake of uniformity with previous papers.

tions of m as a sum of any four squares by the following equations:

$$\lambda(m) = \tfrac{1}{8}\Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)}a, \text{ if } m = 4k+1$$
$$= 0, \text{ if } m = 4k+3;$$
$$P(m) = \tfrac{1}{24}\Sigma^{(r)}(3a^2 - b^2 - c^2 - d^2);$$
$$Q(m) = \tfrac{1}{8}\Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)}a(a^2 - b^2 - c^2 - d^2), \text{ if } m = 4k+1,$$
$$= \tfrac{1}{4}\Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)}bcd, \text{ if } m = 4k+3;$$
$$\Omega(m) = \tfrac{1}{8}\Sigma^{(r)}(a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2a^2d^2 - 2b^2c^2 - 2b^2d^2 - 2c^2d^2);$$

where $\Sigma^{(r)}$ refers to all the representations of m as a sum $a^2+b^2+c^2+d^2$ of four squares, a being uneven and b, c, d, even when $m = 4k+1$, and a being even and b, c, d uneven when $m = 4k+3$; and $s = a+b+c+d$.

Derivation of the coefficient-definitions from the arithmetical definitions, §§ 28–32.

§ 28. The formulæ in § 9 enable us to obtain directly the elliptic-function quantities which have as q-coefficients the functions $\lambda(m)$, $P(m)$, ... defined by these last equations, that is, by means of the representations of m.

For, taking these definitions,

$$8\lambda(4n+1) = \Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)}a$$

= the coefficient of $q^{\frac{1}{4}(4n+1)}$ in $4\Sigma_{-\infty}^{\infty} m' q^{\frac{1}{4}m^2} \times \{\Sigma_{-\infty}^{\infty}(-1)^n q^{\frac{1}{4}\cdot 4n^2}\}^3$,*

that is in $4k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}}\ k'^{\frac{3}{2}}\rho^{\frac{3}{2}} = 4k^{\frac{1}{2}}k'^2\rho^3$.

Since the q-product, when multiplied out, can only contain terms of the form $q^{\frac{1}{4}(4n+1)}$, we have therefore

$$k^{\frac{1}{2}}k'^2\rho^3 = 2\Sigma_0^{\infty}\lambda(4n+1)q^{\frac{1}{4}(4n+1)},$$

which, since $\lambda(4n+3) = 0$, may also be written

$$k^{\frac{1}{2}}k'^2\rho^3 = 2\Sigma_1^{\infty}\lambda(m)q^{\frac{1}{4}m}.$$

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* In general, when n is partitioned into k squares $a^2+b^2+c^2+\ldots$, such a quantity as, say, $\Sigma^{(r)}[a^s b^s c^s d^t e^t f^u]$ is equal to the coefficient of q^n in the expansion of

$$\frac{k!}{3!\,2!\,1!\,(k-6)!}\{\Sigma_{-\infty}^{\infty} n^s q^{n^2}\}^3 \times \{\Sigma_{-\infty}^{\infty} n^t q^{n^2}\}^2 \times \Sigma_{-\infty}^{\infty} n^u q^{n^2} \times \{\Sigma_{-\infty}^{\infty} q^{n^2}\}^{k-6}.$$

The square brackets have the same meaning as in § 26, viz. $[a^s b^s \ldots]$ denotes the sum of all the terms of the type $a^s b^s \ldots$ that can be formed from the k elements, a, b, c, ...

§ 29. In the case of $P(m)$ the definition is

$$24P(m) = \Sigma^{(r)}(3a^2 - b^2 - c^2 - d^2),$$

and, since a is uneven when m is of the form $4k+1$, this equation shows that $24P(4n+1)$ is equal to the coefficient of $q^{\frac{1}{4}(4n+1)}$ in

$$3.4\Sigma_{-\infty}^{\infty} m^2 q^{\frac{1}{4}m^2} \times (\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}.4n^2})^3 - 12\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}m^2} \times \Sigma_{-\infty}^{\infty} (2n)^2 q^{\frac{1}{4}.4n^2} \times (\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}.4n^2})^2,$$

that is, in

$$12(k^{\frac{1}{2}}\rho^{\frac{1}{2}}e.\rho^{\frac{3}{2}} - k^{\frac{1}{2}}\rho^{\frac{1}{2}}.\rho^{\frac{1}{2}}g.\rho) = 12k^{\frac{1}{2}}\rho^2(e-g) = 12k^{\frac{1}{2}}k'^2\rho^4;$$

and, since the expansion of the q-expression can only involve terms of the form $q^{\frac{1}{4}(4n+1)}$, we have therefore

$$k^{\frac{1}{2}}k'^2\rho^4 = 2\Sigma_0^{\infty} P(4n+1)\, q^{\frac{1}{4}(4n+1)}.$$

Similarly we see that $P(4n+3)$ is equal to the coefficient of $q^{\frac{1}{4}(4n+3)}$ in

$$12\Sigma_{-\infty}^{\infty}(2n)^2 q^{\frac{1}{4}.4n^2} \times (\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}m^2})^3 - 12\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}.4n^2} \times \Sigma_{-\infty}^{\infty} m^2 q^{\frac{1}{4}m^2} \times (\Sigma_{-\infty}^{\infty} q^{\frac{1}{4}m^2})^2,$$

that is, in

$$12(\rho^{\frac{1}{2}}g.k^{\frac{3}{2}}\rho^{\frac{3}{2}} - \rho^{\frac{1}{2}}.k^{\frac{1}{2}}\rho^{\frac{1}{2}}e.k\rho) = -12k^{\frac{3}{2}}k'^2\rho^4;$$

whence, we have

$$k^{\frac{3}{2}}k'^2\rho^4 = -2\Sigma_0^{\infty} P(4n+3)\, q^{\frac{1}{4}(4n+3)}.$$

§ 30. Writing for the moment

$$\mu_s = \Sigma_{-\infty}^{\infty} m'^s q^{\frac{1}{4}m^2}, \quad \nu_s = \Sigma_{-\infty}^{\infty} (-1)^n n^s q^{\frac{1}{4}.4n^2},$$

we have, a being uneven and b, c, d even,

$$8Q(4n+1) = \Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)} a(a^2 - b^2 - c^2 - d^2)$$

$$= \text{the coefficient of } q^{\frac{1}{4}(4n+1)} \text{ in } 4\mu_3\nu_0^3 - 12\mu_1\nu_2\nu_0^2,$$

that is, in

$$4k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}}\{3g + (k'^2 - k^2)\rho^2\}\, k'^{\frac{3}{2}}\rho^{\frac{3}{2}} - 12k^{\frac{1}{2}}k'^{\frac{1}{2}}\rho^{\frac{3}{2}}.k'^{\frac{1}{2}}\rho^{\frac{1}{2}}i.k'\rho$$

$$= 4k^{\frac{1}{2}}k'^2\rho^3\{3(g-i) + (k'^2 - k^2)\rho^2\} = 4k^{\frac{1}{2}}k'^2(1+k^2)\rho^5.$$

The q-expression, when expanded, consists only of terms of the form $q^{\frac{1}{4}(4n+1)}$, so that we have

$$k^{\frac{1}{2}}k'^{2}(1+k^{2})\rho^{5} = 2\Sigma_{0}^{\infty} Q(4n+1)\, q^{\frac{1}{4}(4n+1)}.$$

Treating in the same manner the case of a even and b, c, d uneven, we have

$$4Q(4n+3) = \Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)} bcd$$

= the coefficient of $q^{\frac{1}{4}(4n+3)}$ in $-4\mu_1{}^3\nu_0$,* that is, in $-4k^{\frac{3}{2}}k'^{2}\rho^{5}$; and therefore

$$k^{\frac{3}{2}}k'^{2}\rho^{5} = -\Sigma_{0}^{\infty} Q(4n+3)\, q^{\frac{1}{4}(4n+3)}.$$

§ 31. The expression for $\Omega(m)$ is the same, whether m is of the form $4k+1$ or $4k+3$, viz.

$$8\Omega(m) = \Sigma^{(r)}\{a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2 - 2b^2c^2-2b^2d^2-2c^2d^2\}.$$

Putting for brevity

$$M_s = \Sigma_{-\infty}^{\infty} m^s q^{\frac{1}{4}m^2}, \quad N_s = \Sigma_{-\infty}(2n)^s q^{\frac{1}{4}\cdot 4n^2},$$

we see that $8\Omega(4n+1)$ is equal to the coefficient of $q^{\frac{1}{4}(4n+1)}$ in the expansion of the q-expression

$$4(M_4N_0^3 + 3N_4N_0^2M_0 - 6M_2N_2N_0^2 - 6N_2^2N_0M_0),$$

and the value of this expression on reduction is found to be $4k^{\frac{1}{2}}k'^{2}(1+3k^2)\rho^6$.

Since the q-expression, when expanded, can contain only terms of the form $q^{\frac{1}{4}(4n+1)}$, we find therefore

$$k^{\frac{1}{2}}k'^{2}(1+3k^2)\rho^{6} = 2\Sigma_{0}^{\infty}\Omega(4n+1)\, q^{\frac{1}{4}(4n+1)}.$$

Similarly $8\Omega(4n+3)$ is equal to the coefficient of $q^{\frac{1}{4}(4n+3)}$ in

$$4(N_4M_0^3 + 3M_4M_0^2N_0 - 6N_2M_2M_0^2 - 6M_2^2M_0N_0),$$

that is, in $-4k^{\frac{3}{2}}k'^{2}(3+k^2)\rho^6$; whence

$$k^{\frac{3}{2}}k'^{2}(3+k^2)\rho^{6} = -2\Sigma_{0}^{\infty}\Omega(4n+3)\, q^{\frac{1}{4}(4n+3)}.$$

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* The minus sign is due to the fact that in the product $\mu_1{}^3\nu_0$ the term bcd has the sign $(-1)^{\frac{1}{2}\alpha+\frac{1}{2}(b-1)+\frac{1}{2}(c-1)+\frac{1}{2}(d-1)} = -(-1)^{\frac{1}{2}(s-1)}$.

§ 32. Thus, from the arithmetical definitions of $\lambda(m)$, ... in terms of the representations of m as a sum of four squares, we have derived the expansions:

$$k^{\frac{1}{2}}k'^{2}\rho^{3} = 2\Sigma_{0}^{\infty}\lambda(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{2}\rho^{4} = 2\Sigma_{0}^{\infty}P(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{2}(1+k^{2})\rho^{5} = 2\Sigma_{0}^{\infty}Q(4n+1)\,q^{\frac{1}{4}(4n+1)},$$
$$k^{\frac{1}{2}}k'^{2}(1+3k^{2})\rho^{6} = 2\Sigma_{0}^{\infty}\Omega(4n+1)\,q^{\frac{1}{4}(4n+1)},$$

and
$$k^{\frac{3}{2}}k'^{2}\rho^{4} = -2\Sigma_{0}^{\infty}P(4n+3)\,q^{\frac{1}{4}(4n+3)},$$
$$k^{\frac{3}{2}}k'^{2}\rho^{5} = -\Sigma_{0}^{\infty}Q(4n+3)\,q^{\frac{1}{4}(4n+3)},$$
$$k^{\frac{3}{2}}k'^{2}(3+k^{2})\rho^{6} = -2\Sigma_{0}^{\infty}\Omega(4n+3)\,q^{\frac{1}{4}(4n+3)}.$$

From these equations, by changing q into q^{2}, q into $-q$, adding and substituting, &c., we may obtain all the expansions involving $\lambda(m)$, ... given on pp. 342, 346, 349, 350, and 353 of vol. XXXVI.

The function $\Omega(n)$, § 33.

§ 33. The arithmetical definition of the function $\Omega(m)$, by means of the representations of m as a sum of four squares, is equally appropriate to the case in which the argument is even, viz. we may define $\Omega(m)$ for any value of n by the equation

$$\Omega(n) = \tfrac{1}{8}\Sigma^{(r)}\{[a^{4}] - 2[a^{2}b^{2}]\},$$

where $a^{2}+b^{2}+c^{2}+d^{2}$ is any partition of n into four squares.

Adopting this definition it is evident that $8\Omega(n)$ is the coefficient of q^{n} in

$$4\Sigma_{-\infty}^{\infty}n^{4}q^{n^{2}} \times (\Sigma_{-\infty}^{\infty}q^{n^{2}})^{3} - 2.6\,(\Sigma_{-\infty}^{\infty}n^{2}q^{n^{2}})^{2}(\Sigma_{-\infty}^{\infty}q^{n^{2}})^{2},$$

that is, in

$$\tfrac{1}{24}\{4\rho^{\frac{1}{2}}(3g^{2}+2k^{2}k'^{2}\rho^{4})\,\rho^{\frac{3}{2}} - 12\rho g^{2}.\rho\} = \tfrac{1}{2}k^{2}k'^{2}\rho^{6},$$

whence we have

$$k^{2}k'^{2}\rho^{6} = 16\Sigma_{1}^{\infty}\Omega(n)\,q^{n}.*$$

Changing the sign of q, the left-hand side becomes $-k^{2}k'^{2}\rho^{6}$, which shows that $\Omega(n) = 0$ when n is even.

Theorem relating to representation by four squares, § 34.

§ 34. The result just obtained, viz. that if N be any even number, $\Omega(N) = 0$ shows that, for any even number N,

$$\Sigma^{(r)}\{[a^{4}] - 2[a^{2}b^{2}]\} = 0.$$

* This equation was taken as the definition of $\Omega(n)$ in vol. XXXVIII., p. 201. Previously I had only defined $\Omega(m)$.

We may also write this equation in the form

$$\Sigma^{(r)}\{2[a^4]-N^2\}=0,$$

whence, since the number of representations of N is equal to $24\Delta(N)$, we have

$$2\Sigma^{(r)}[a^4]=N^2.24\Delta(N),$$

that is, $$\Sigma^{(r)}(a^4+b^4+c^4+d^4)=12N^2\Delta(N),$$

which is a curious theorem giving the value of the sum of the fourth powers of all the elements of the representations of N as a sum of four squares.

As an example take $N=10$. The partitions are

$$3^2+1^2+0^2+0^2 \text{ and } 2^2+2^2+1^2+1^2,$$

which give rise to 48 and 96 representations respectively. Thus the theorem gives

$$48(3^4+1^4+0^4+0^4)+96(2^4+2^4+1^4+1^4)=12\times100\times6.$$

The Function $\Theta(n)$, §§ 35–49.

Derivation of arithmetical definitions of $\Theta(n)$ from the coefficient-definition, §§ 35–42.

§ 35. For expressing the number of representations of n as a sum of sixteen squares use has been made of the function $\Theta(n)$ defined by the equation

$$k^2k'^4\rho^8=\Sigma_1^\infty\Theta(n)\,q^n.*$$

It will now be shown that from the formulæ of § 9 we may derive an arithmetical definition of $\Theta(n)$ depending upon the representations of n as a sum of four squares.

Denoting, as in § 31, $\Sigma_{-\infty}^{\infty}(2n)^s q^{n^2}$ by N_s, we have

$$N_2^3N_0=\rho^2g^3,\quad N_4N_2N_0^2=\rho^2(3g^3+2gk^2k'^2\rho^4),$$
$$N_6N_0^3=\rho^2\{15g^3+30gk^2k'^2\rho^4+8k^2k'^2(k'^2-k^2)\rho^6\},$$

whence we find

$$8k^2k'^2(k'^2-k^2)\rho^8=N_6N_0^3-15N_4N_2N_0^2+30N_2^3N_0.$$

Denoting any partition of n by $a^2+b^2+c^2+d^2$ the coefficient of q^n on the right-hand side

$$=2^6\{\tfrac{1}{4}\Sigma^{(r)}[a^6]-\tfrac{15}{2}\Sigma^{(r)}[a^4b^2]+\tfrac{30}{4}\Sigma^{(r)}[a^2b^2c^2]\},$$

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* Vol. XXXVIII., p. 193.

and therefore the coefficient of q^n in the expansion of $k^2k'^2(k'^2-k^2)\rho^8$

$$=2\Sigma^{(r)}\{[a^6]-5[a^4b^2]+30[a^2b^2c^2].$$

Now $$k^2k'^4\rho^8= 16\Sigma_1^\infty\Theta(n)q^n,$$

and $$k^4k'^2\rho^8=-32\Sigma_1^\infty\Theta(2n)q^{2n},$$

whence

$$(k^2k'^4-k^4k'^2)\rho^8=16\Sigma_1^\infty\Theta(m)q^m+48\Sigma_1^\infty\Theta(2n)q^{2n}.$$

Thus, putting

$$A=\Sigma^{(r)}\{[a^6]-5[a^4b^2]+30[a^2b^2c^2]\},$$

we have $\Theta(n)=\frac{1}{8}A$ or $\frac{1}{24}A$ according as n is uneven or even.

Taking as a numerical example $n=13$, the partitions of n into four squares are $3^2+2^2+0^2+0^2$ and $2^2+2^2+2^2+1^2$ which give rise respectively to 48 and 64 representations. Thus

$$\tfrac{1}{8}A=6\{3^6+2^6-5(3^42^2+2^43^2)+30.0\}$$

$$+8\{3.2^6+1^6-5(3.2^41^2+3.1^42^2+6.2^42^2)+30(3.2^22^21^2+2^22^22^2)\}$$

$$=6(793-2340)+8(193-2220+3360)=-9282+10664=1382,$$

which is the value of $\Theta(13)$ (vol. XXXVIII., p. 198).

Taking as another example $n=14$, there is but one partition $3^2+2^2+1^2+0^2$, giving rise to 192 representations. Thus

$$\tfrac{1}{24}A=8\{3^6+2^6+1^6-5(3^42^2+2^43^2+3^41^2+1^43^2+2^41^2+1^42^2)+30.3^2.2^2.1^2\}$$

$$=8(794-2890+1080)=-8\times1016=-8128,$$

which is the value of $\Theta(14)$ (vol. XXXVIII., p. 198).

§ 36. Since $a^2+b^2+c^2+d^2=n$, we have

$$\Sigma^{(r)}\{[a^6]+3[a^4b^2]+6[a^2b^2c^2]=\Sigma^{(r)}n^3,$$

and $\Sigma^{(r)}n^3=8n^3\Delta(n)$ or $24n^3\Delta(n)$ according as n is uneven or even (vol. XXXVIII., p. 8).

We may therefore remove from the value of A any one of the three terms in $[a^6]$, $[a^4b^2]$, or $[a^2b^2c^2]$. For example, if we remove the term in $[a^6]$, we have

$$A=\Sigma^{(r)}n^3-8\Sigma^{(r)}[a^4b^2]+24\Sigma^{(r)}[a^2b^2c^2],$$

which $$=8\{\ n^3\Delta(n)-\Sigma^{(r)}[a^4b^2]+3\Sigma^{(r)}[a^2b^2c^2]\},$$

or $$=8\{3n^3\Delta(n)-\Sigma^{(r)}[a^4b^2]+3\Sigma^{(r)}[a^2b^2c^2]\},$$

according as n is uneven or even.

It is, however, more convenient to remove the term in $[a^4b^2]$ as being the most complicated of the three: we thus find

$$A = \tfrac{1}{3}\{8\Sigma^{(r)}[a^6] + 120\Sigma^{(r)}[a^2b^2c^2] - 5\Sigma^{(r)}n^3\},$$

and therefore, m being any uneven, and N any even, number,

$$\Theta(m) = \tfrac{1}{3}\{\Sigma^{(r)}[a^6] + 15\Sigma^{(r)}[a^2b^2c^2] - 5m^3\Delta(m)\},$$
$$\Theta(N) = \tfrac{1}{9}\{\Sigma^{(r)}[a^6] + 15\Sigma^{(r)}[a^2b^2c^2] - 15N^3\Delta(N)\}.$$

§ 37. As an example take $m = 45$. The partitions of 45, with the numbers of representations to which they give rise, are

$$6^2+2^2+2^2+1^2\,(192), \quad 6^2+3^2+0^2+0^2\,(48),$$
$$5^2+4^2+2^2+0^2\,(192), \quad 4^2+4^2+3^2+2^2\,(192),$$

and therefore

$$\begin{aligned}\Theta(45) &= \tfrac{1}{3}\{\Sigma^{(r)}[a^6] + 15\Sigma^{(r)}[a^2b^2c^2] - 5.45^3.\Delta(45)\}\\
&= \tfrac{1}{3}\{192(6^6+2.2^6+1^6) + 48(6^6+3^6) + 192(5^6+4^6+2^6)\\
&\quad + 192(2.4^6+3^6+2^6)\}\\
&\quad + 5\{192(6^2 2^2 2^2 + 2.6^2 2^2 1^2 + 2^2 2^2 1^2) + 48.0 + 192.5^2 4^2 2^2\\
&\quad + 192(4^2 4^2 3^2 + 4^2 4^2 2^2 + 2.4^2 3^2 2^2)\} - \tfrac{5}{3}.45^3.78\\
&= \tfrac{1}{3}.16781040 + 5.1336320 - 11846250\\
&= 429030.\end{aligned}$$

Since $\Theta(5) = -210$ and $\Theta(9) = -2043$ (vol. XXXVIII., p. 198), we therefore find that

$$\Theta(45) = \Theta(5)\,\Theta(9)\ *$$

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* When calculating the table $\Theta(m)$ in vol. XXXVIII., p. 198 the only means at my disposal were afforded by two recurring formulæ, neither of which was very satisfactory. The formula obtained in the text, viz.

$$\Theta(m) = \tfrac{1}{3}\{\Sigma^{(r)}[a^6] + 15\Sigma^{(r)}[a^2b^2c^2] - 5m^3\Delta(m)\}$$

enables us to calculate any isolated value of $\Theta(m)$.

When writing the paper in vol. XXXVIII. I was only able to verify (on account of the small extent of the table) the formula $\Theta(m_1m_2) = \Theta(m_1)\,\Theta(m_2)$ for $m_1, m_2 = 3, 5$ and 3, 7, one of the factors thus being of the form $4k+3$ in both cases (p. 198). In the case of $m_1, m_2 = 5, 9$, which is proved in the text, both factors are of the form $4k+1$. We have therefore grounds for inferring that the formula is true generally when m_1 and m_2 are prime to each other, and it probably admits of being proved by the methods used in the paper in vol. XXXVII., pp. 36–48.

The results obtained in §§ 34–40 of the present paper have been referred to on p. 198 (note) and pp. 234–5 of vol. XXXVIII.

§ 38. We can express $\Theta(m)$ also by means of the compositions of $4m$ as a sum of four uneven squares; for, proceeding as in § 35 and using, as before, M_s to denote $\Sigma_{-\infty}^{\infty} m^s q^{4m^2}$, we have

$$M_2^3 M_0 = k^2\rho^2 e^3, \quad M_4 M_2 M_0^3 = k^2\rho^2(3e^3 - 2ek'^2\rho^4),$$

$$M_6 M_0^3 = k^2\rho^2\{15e^2 - 30ek'^2\rho^4 + 8k'^2(1+k'^2)\rho^6\},$$

whence

$$8k^2k'^2(1+k'^2)\rho^8 = M_6 M_0^3 - 15 M_4 M_2 M_0^2 + 30 M_2^3 M_0.$$

Now, if a number is the sum of four uneven squares it must be of the form $8k+4=4m$, m being any uneven number, and therefore the expansion of the quantity $M_6 M_0^3 -$, &c. can only involve uneven powers of q.*

Let $\alpha^2+\beta^2+\gamma^2+\delta^2$ be any partition of $4m$ into four uneven squares, then, reasoning as in § 35, we see that the coefficient of q^m in $8k^2k'^2(1+k'^2)\rho^8$ is equal to

$$\tfrac{1}{4}\Sigma^{(\alpha)}\{[\alpha^6] - 5[\alpha^4\beta^2] + 30[\alpha^2\beta^2\gamma^2]\},$$

and therefore, passing from representations to compositions, the coefficient of q^m in $k^2k'^2(1+k'^2)\rho^8$ is equal to

$$\tfrac{1}{2}\Sigma^{(c)}\{[\alpha^6] - 5[\alpha^4\beta^2] + 30[\alpha^2\beta^2\gamma^2]\}.$$

§ 39. Now $\quad k^2k'^4\rho^8 = 16\Sigma_1^{\infty}\Theta(n)\,q^n,$

and $\quad k^2k'^2\rho^8 = 16\Sigma_1^{\infty}\Theta(-1)^{n-1}\Theta(n)\,q^n,$

so that $\quad (k^2k'^2 + k^2k'^4)\rho^8 = 32\Sigma_1^{\infty}\Theta(m)\,q^m,$

and we therefore find

$$\Theta(m) = \tfrac{1}{64}\Sigma^{(c)}\{[\alpha^6] - 5[\alpha^4\beta^2] + 30[\alpha^2\beta^2\gamma^2]\}.$$

§ 40. We can (as in § 36) get rid of any one of these three terms; for, since $\alpha^2+\beta^2+\gamma^2+\delta^2=4m$, we have

$$\Sigma^{(c)}\{[\alpha^6] + 3[\alpha^4\beta^2] + 6[\alpha^2\beta^2\gamma^2] = \Sigma^{(c)} 64m^3 = 64m^3\Delta(m).$$

Substituting therefore for the term $\Sigma^{(c)}[\alpha^4\beta^2]$, we find

$$\Theta(m) = \tfrac{1}{24}\{\Sigma^{(c)}[\alpha^6] + 15\Sigma^{(c)}[\alpha^2\beta^2\gamma^2] - 40m^3\Delta(m)\}.$$

---

* It is evident also that the expansion of $(k^2k'^2+k^2k'^4)\rho^8$ can only involve uneven powers of q, for, changing the sign of q, this quantity becomes $(-k^2k'^4-k^2k'^2)\rho^8$, that is, it becomes the same expression with the sign changed.

§ 41. Taking the same example as before, viz, $m=45$, we have $4m=180$, and the partitions of $4m$ into four uneven squares, with the numbers of their compositions, are

$13^2+3^2+1^2+1^2(12)$, $11^2+7^2+3^2+1^2(24)$, $11^2+5^2+5^2+3^2(12)$,

$9^2+9^2+3^2+3^2(6)$, $9^2+7^2+7^2+1^2(12)$, $9^2+7^2+5^2+5^2(12)$.

Thus

$$\Sigma^{(c)}[\alpha^6]=12(13^6+3^6+2.1^6)+24(11^6+7^6+3^6+1^6)+12(11^6+2.5^6+3^6)$$
$$+6(2.9^6+2.3^6)+12(9^6+2.7^6+1^6)+12(9^6+7^6+2.5^6)$$
$$=148682520.$$

$$\Sigma^{(c)}[\alpha^2\beta^2\gamma^2]=12\,(2.13^2 3^2 1^2+13^2 1^2 1^2+3^2 1^2 1^2)+\&c.$$
$$=9728280,$$

and $$40m^3\Delta\,(m)=40.45^3.78=284310000,$$

whence, by the formula in § 40,

$$\Theta\,(45)=\tfrac{1}{24}\,(148682520+15\times 9728280-284310000)$$
$$=\tfrac{1}{24}.10296720=429030,$$

as before (§ 37).

§ 42. Doubtless the two values of $\Theta\,(m)$ which have been found in §§ 35 and 39, viz.

$$\Theta\,(m)=\tfrac{1}{8}\Sigma^{(r)}\{[\alpha^6]-5\,[\alpha^4b^2]+30\,[\alpha^2b^2c^2]\}\quad(\S\,35),$$

$$\Theta\,(m)=\tfrac{1}{64}\Sigma^{(c)}\{[\alpha^6]-5\,[\alpha^4\beta^2]+30\,[\alpha^2\beta^2\gamma^2]\}\quad(\S\,39)$$

can be transformed into one another by the same formulæ of transformation as those by which the two values of Ω, viz.

$$\Omega\,(m)=\tfrac{1}{8}\Sigma^{(r)}\{[\alpha^4]-2\,[\alpha^2b^2]\}\quad\text{(vol. XXXVI., p. 334)},$$

$$\Omega\,(m)=\Sigma^{(c)}\alpha'\beta'\gamma'\delta'\quad\text{(vol. XXXVI., p. 323)}$$

were connected in vol. XXXVI., pp. 331–2.

Other forms of the values of $\Theta\,(m)$, §§ 43–46.

§ 43. We know that

$$\Sigma^{(r)}\{[\alpha^4]-2\,[\alpha^2b^2]\}=8\Omega\,(n),$$

where $\Omega\,(n)=0$, when n is even.

Multiplying $[a^4]-2[a^2b^2]$ by $[a^2]$, we obtain as product $[a^6]-[a^4b^2]-6[a^2b^2c^2]$, and therefore

$$\Sigma^{(r)}\{[a^6]-[a^4b^2]-6[a^2b^2c^2]\}=8n\Omega(n),$$

or, putting for brevity

$$A=\Sigma^{(r)}[a^6],\ B=\Sigma^{(r)}[a^4b^2],\ C=\Sigma^{(r)}[a^2b^2c^2],$$

this equation is

$$A-B-6C=8n\Omega(n).$$

We have also, from § 36,

$$A+3B+6C=\Sigma^{(r)}n^3,$$

and, from § 35,

$$A-5B+30C=8\{2+(-1)^n\}\Theta(n).$$

§ 44. Taking first the case of n uneven, the three equations are

$$A-B-6C=8m\Omega(m),$$
$$A+3B+6C=8m^3\Delta(m),$$
$$A-5B+30C=8\Theta(m),$$

whence, solving for A, we find

$$2A=\Theta(m)+10m\Omega(m)+5m^3\Delta(m),$$

giving

(i) $$\Theta(m)=2\Sigma^{(r)}[a^6]-10m\Omega(m)-5m^3\Delta(m).$$

If we solve for C, we find

$$6C=\Theta(m)-2m\Omega(m)+m^3\Delta(m),$$

giving

(ii) $$\Theta(m)=6\Sigma^{(r)}[a^2b^2c^2]+2m\Omega(m)-m^3\Delta(m).$$

§ 45. Taking now the case of n even, $=N$, the three equations are

$$A-B-6C=0,$$
$$A+3B+6C=24N^3\Delta(N),$$
$$A-5B+30C=24\Theta(N);$$

whence, solving, we find

$$2A = 3\Theta(N) + 15N^3\Delta(N),$$
$$2C = \Theta(N) + N^3\Delta(N),$$

giving

(iii) $\Theta(N) = \frac{2}{3}\Sigma^{(r)}[a^6] - 5N^3\Delta(N)$,

and (iv) $\Theta(N) = 2\Sigma^{(r)}[a^2b^2c^2] - N^3\Delta(N)$.

The formulæ (i) and (ii) express $\Theta(m)$ in terms of $\Sigma^{(r)}[a^6]$ or $\Sigma^{(r)}[a^2b^2c^2]$ only, if $\Omega(m)$ is already known,* but even then they possess but little advantage over the formula for $\Theta(m)$ in § 36. It is otherwise, however, with the formulæ (iii) and (iv) for $\Theta(N)$, which are simpler than the formula for $\Theta(N)$ in § 36.

§ 46. Since $\Theta(2m) = -8\Theta(m)$, the formulæ (iii) and (iv) afford expressions for $\Theta(m)$ in terms of the representations of $2m$ by four squares, viz. if $a_1^2 + b_1^2 + c_1^2 + d_1^2$ is any partition of $2m$ into any four squares, then

$$\Theta(m) = -\tfrac{1}{12}\Sigma^{(r)}[a_1^6] + 5m^3\Delta(m),$$

and $$\Theta(m) = -\tfrac{1}{4}\Sigma^{(r)}[a_1^2b_1^2c_1^2] + m^3\Delta(m).$$

Other formulæ for $\Theta(m)$ are given in § 49.

Representations of m, 2m, and 4m, §§ 47–49.

§ 47. The partitions into any four squares of any number of the form $4N$, where N is even, consist of even squares only, and are derivable from those of N by quadrupling all the squares in the partitions of N. In connection therefore with the representations of even numbers by four squares, we need only consider numbers of the forms $2m$ and $4m$, m being uneven. In the former case the partitions consist of two uneven and two even squares; in the latter case they consist (i) wholly of even squares (derivable from the partitions of m into four squares by quadrupling the squares), and (ii) wholly of uneven squares.

§ 48. Let a, b, c, d refer to any partition of m into four squares, a_1, b_1, c_1, d_1 to any partition of $2m$, a_2, b_2, c_2, d_2 to any

* A table of $\Omega(m)$ up to $m = 51$ was given in vol. XXXVIII., p. 56.

partition of $4m$, and α, β, γ, δ to any partition of $4m$ into uneven squares. Then, from § 35, we have

$$\Theta(m) = \tfrac{1}{3}\Sigma^{(r)}\{[a^6] + 15[a^2b^2c^2]\} - \tfrac{5}{3}m^3\Delta(m),$$
$$\Theta(2m) = \tfrac{1}{9}\Sigma^{(r)}\{[a_1^6] + 15[a_1^2b_1^2c_1^2]\} - \tfrac{5}{3}8m^3\Delta(m),$$
$$\Theta(4m) = \tfrac{1}{9}\Sigma^{(r)}\{[a_2^6] + 15[a_2^2b_2^2c_2^2]\} - \tfrac{5}{3}64m^3\Delta(m).$$

Since $\Theta(2m) = -8\Theta(m)$ and $\Theta(4m) = 64\Theta(m)$, these equations give

$$\Sigma^{(r)}\{[a_1^6] + 15[a_1^2b_1^2c_1^2]\} = -24\Sigma^{(r)}\{[a^6] + 15[a^2b^2c^2]\} + 240m^3\Delta(m),$$

and $$\Sigma^{(r)}\{[a_2^6] + 15[a_2^2b_2^2c_2^2]\} = 192\Sigma^{(r)}\{[a^6] + 15[a^2b^2c^2]\}.$$

This last equation shows that

$$\Sigma^{(r)}\{[\alpha^6] + 15[\alpha^2\beta^2\gamma^2]\} = 128\Sigma^{(r)}\{[a^6] + 15[a^2b^2c^2]\}.$$

§ 49. The formulæ (iii) and (iv) of § 45 show that

$$\Theta(2m) = \tfrac{2}{3}\Sigma^{(r)}[a_1^6] - 5.8m^3\Delta(m),$$
$$\Theta(4m) = \tfrac{2}{3}\Sigma^{(r)}[a_2^6] - 5.64m^3\Delta(m),$$

and

$$\Theta(2m) = 2\Sigma^{(r)}[a_1^2b_1^2c_1^2] - 8m^3\Delta(m),$$
$$\Theta(4m) = 2\Sigma^{(r)}[a_2^2b_2^2c_2^2] - 64m^3\Delta(m);$$

from which we may derive the relations

$$\Sigma^{(r)}\{[a_2^6] + 8[a_1^6]\} = 960m^3\Delta(m),$$
$$\Sigma^{(r)}\{[a_2^2b_2^2c_2^2] + 8[a_1^2b_1^2c_1^2]\} = 64m^3\Delta(m),$$

which may also be written

$$\Sigma^{(r)}\{64[a^6] + 8[a_1^6] + [\alpha^6]\} = 960m^3\Delta(m),$$
$$\Sigma^{(r)}\{64[a^2b^2c^2] + 8[a_1^2b_1^2c_1^2] + [\alpha^2\beta^2\gamma^2]\} = 64m^3\Delta(m).$$

It may be remarked that we have also

$$\Theta(m) = \tfrac{1}{192}\Sigma^{(r)}\{[a_2^6] - 8[a_1^6]\}$$
$$= \tfrac{1}{64}\Sigma^{(r)}\{[a_2^2b_2^2c_2^2] - 8[a_1^2b_1^2c_1^2]\}.$$

Expressions for $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$ as sums of powers of complex numbers, §§ 50–69.

§ 50. I have obtained expressions for $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$, as the sum of the first, second, third, and fourth powers of certain complex numbers derived from the representations

of m by four squares. These complex numbers (which I have termed primary derivates of the representations) are defined and considered in the next six sections.

Derivates of representations and partitions, §§ 51–53.

§ 51. 1°. First consider a partition $a^2+b^2+c^2+d^2$, in which a is uneven and b, c, d even, or in which a is even and b, c, d uneven, a, b, c, d being all different and none of them zero. This partition gives rise to 384 representations in 96 of which a stands first. Restricting ourselves to representations in which a stands first, and denoting by (a, b, c, d) the representation $a^2+b^2+c^2+d^2$, so that for example $(a, -c, b, d)$ denotes the representation $a^2+(-c)^2+b^2+d^2$, suppose that all the 96 representations of m in which a stands first are written down, and that from them we form 96 complex expressions by multiplying the first element of each representation by 1, the second by i, the third by i^2, and the fourth by i^3 (that is, the four elements are multiplied by $1, i, -1, -i$). It will be found that in this manner we obtain, each twice over, the following 24 complex numbers.

$$\begin{array}{lll}
a+ib+c+id, & a+ib+ic+d, & a+b+ic+id,\\
a-ib+c+id, & a-ib+ic+d, & a-b+ic+id,\\
a+ib-c+id, & a+ib-ic+d, & a+b-ic+id,\\
a+ib+c-id, & a+ib+ic-d, & a+b+ic-id,\\
a-ib-c+id, & a-ib-ic+d, & a-b-ic+id,\\
a-ib+c-id, & a-ib+ic-d, & a-b+ic-id,\\
a+ib-c-id, & a+ib-ic-d, & a+b-ic-id,\\
a-ib-c-id, & a-ib-ic-d, & a-b-ic-id,
\end{array}$$

in which the sign of a is positive, and also (each twice over) 24 complex expressions which differ from these only by a change of sign of the whole expression.

Each of these complex expressions may be conveniently called a complex derivate, or simply *a derivate of the representation* from which it is derived, and the whole system may be called the *derivates of the partition* from which they are derived.*

* In writing down the system of derivates I have put in the first line those in which all the signs are positive. The second line is obtained by changing the sign of b, the third of c, the fourth of d, the fifth of b and c, the sixth of b and d, the seventh of c and d, and the eighth of b, c, and d.

2°. Consider now a partition of the form $a^2+b^2+c^2+c^2$, that is, one in which two squares are equal and there is no zero square.

By writing down all the representations in which a stands first, and forming from them the derivates, we obtain the following system of 48 derivates, viz.

$$\begin{aligned} &a+b, \quad a+ib+c+ic,\\ &a-b, \quad a-ib+c+ic,\\ &\qquad a+ib-c+ic,\\ &\qquad a+ib+c-ic,\\ &\qquad a-ib-c+ic,\\ &\qquad a-ib+c-ic,\\ &\qquad a+ib-c-ic,\\ &\qquad a-ib-c-ic, \end{aligned}$$

occurring twice;

$$\begin{aligned} &a+b+2ic, \quad a-b+2ic,\\ &a+b-2ic, \quad a-b-2ic, \end{aligned}$$

occurring once; and 24 others differing from these only by a change of sign of the whole derivate.

3°. If three squares are equal, so that the partition is $a^2+b^2+b^2+b^2$, in which neither a nor b is zero, the derivates are found to be

$$a+b, \quad a-b,$$

occurring twice;

$$\begin{aligned} &a+b+2ib, \quad a-b+2ib,\\ &a+b-2ib, \quad a-b-2ib, \end{aligned}$$

occurring once; and 8 others differing from these only by a change of sign of the whole derivate.*

4°. A partition of the form $a^2+b^2+c^2+0^2$, where a, b, c are different and non-zero, produces 48 derivates, viz.

$$\begin{aligned} &a+ib+c, \quad a+ib+ic, \quad a+b+ic,\\ &a-ib+c, \quad a-ib+ic, \quad a-b+ic,\\ &a+ib-c, \quad a+ib-ic, \quad a+b-ic,\\ &a-ib-c, \quad a-ib-ic, \quad a-b-ic, \end{aligned}$$

* The case of all four squares equal does not arise, as we are only concerned with the partitions of uneven numbers.

occurring twice; and 24 others differing from these only by a change of sign of the whole derivate.

5°. A partition of the form $a^2+b^2+b^2+0^2$, where a and b are different and non-zero, produces 24 derivates, viz.

$$a,\quad a+b+ib,$$
$$a-b+ib,$$
$$a+b-ib,$$
$$a-b-ib,$$

occurring twice;

$$a+2ib,\quad a-2ib,$$

occurring once; and 12 others differing from these only by a change of sign of the whole derivate.

6°. A partition of the form $a^2+b^2+0^2+0^2$, where a and b are different and non-zero, produces 12 derivates, viz.

$a+ib$, $a-ib$ occurring twice;

$a+b$, $a-b$ occurring once;

and six others differing from these only by a change of sign of the whole derivate.

7°. A partition of the form $a^2+0^2+0^2+0^2$ produces only two derivates a and $-a$.

§ 52. If in the derivate 1° we put $d=c$, we obtain 96 expressions which are identical in groups of four; if we discard half of these, leaving 48 which are identical in groups of two, we obtain the derivates in 2°.

Similarly, if in the derivates in 1° we put $d=c=b$, the 96 expressions so obtained are identical in groups of 12 and of 6. If we retain only two from each group of 12, and one from each group of 6, we obtain the derivates 3°.

With respect to 4°, 5°, 6°, 7°, in which zeros occur in the partitions, the derivates are found to be derivable from those in 1° by taking respectively $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, and $\frac{1}{48}$ of the members of each of the groups of identical expressions which result when the changes required to transform the partition in 1° to those in 4°, 5°, 6°, and 7° have been made.*

* These results might have been predicted generally; but it seemed safer to actually work out separately the derivates for each form of partition. It also seemed to be of interest to exhibit the actual derivates in the different cases.

§ 53. It follows therefore that if the sum of the r^{th} powers of the 96 derivates in 1° is denoted by $A_r(a, b, c, d)$, then the sum of the r^{th} powers of the derivates

$$\begin{aligned}
\text{in } 2^\circ &= \tfrac{1}{2}A_r(a, b, c, c),\\
,, \ 3^\circ &= \tfrac{1}{6}A_r(a, b, b, b),\\
,, \ 4^\circ &= \tfrac{1}{2}A_r(a, b, c, 0),\\
,, \ 5^\circ &= \tfrac{1}{4}A_r(a, b, b, 0),\\
,, \ 6^\circ &= \tfrac{1}{8}A_r(a, b, 0, 0),\\
,, \ 7^\circ &= \tfrac{1}{48}A_r(a, 0, 0, 0),
\end{aligned}$$

where $A_r(a, b, c, c)$ denotes what $A_r(a, b, c, d)$ becomes when d is put equal to c, and so on.

Primary derivates, §§ 54–55.

§ 54. The derivates are ordinary complex numbers, and from them I select those (one-half in number) which are primary according to Gauss's definition. Denoting by s the sum of the elements of the partition (each element having the positive sign), then, if m is of the form $4k+1$ (in which case a is uneven and b, c, d are even), the primary derivates of the partition are those in which the sign of a is positive, each multiplied by $(-1)^{\frac{1}{2}(s-1)}$.

But, if m is of the form $4k+3$ (in which a is even and b, c, d uneven), one-half of the primary derivates consist of derivates in which the sign of a is positive and the other half of derivates in which the sign of a is negative, all multiplied by $(-1)^{\frac{1}{2}(s-1)}$; or, stated otherwise, if we derive the primary derivates entirely from those in which the sign of a is positive, half of the primary derivates are obtained by multiplying half of them by $(-1)^{\frac{1}{2}(s-1)}$, and the other half by multiplying the other half by $-(-1)^{\frac{1}{2}(s-1)}$. Thus, taking the derivates (with sign of a positive) written down in 1° (§ 51) the primary derivates consist of those in which all four signs are positive, and those in which two are positive and two negative, multiplied by $(-1)^{\frac{1}{2}(s-1)}$, and of those in which three signs are positive and one negative, and those in which three are negative and one positive, multiplied by $-(-1)^{\frac{1}{2}(s-1)}$.

55. It is evident that results similar to those in § 53 hold good also in the case of sums of powers of primary derivates.

Value of $\lambda(m)$ *in terms of primary derivates*, §§ 56–58.

§ 56. It it now convenient to distinguish by means of suffixes, between the different forms of partitions in 1°, 2°, ..., 7°

(§ 51), viz. the forms of the partitions to which the suffixes relate are:

$$\begin{aligned}
1^\circ,&\quad a_1^2+b_1^2+c_1^2+d_1^2,\\
2^\circ,&\quad a_2^2+b_2^2+c_2^2+c_2^2,\\
3^\circ,&\quad a_3^2+b_3^2+b_3^2+b_3^2,\\
4^\circ,&\quad a_4^2+b_4^2+c_4^2+0^2,\\
5^\circ,&\quad a_5^2+b_5^2+b_5^2+0^2,\\
6^\circ,&\quad a_6^2+b_6^2+0^2+0^2,\\
7^\circ,&\quad a_7^2+0^2+0^2+0^2.
\end{aligned}$$

§ 57. If m is of the form $4k+1$,

$$\lambda(m)=\tfrac{1}{8}\Sigma^{(r)}(-1)^{\frac{1}{4}(s-1)}a \quad (\S\,27),$$

that is, separating the partitions of the different forms,

$$\begin{aligned}
\lambda(m)=48\Sigma(-1)^{\frac{1}{2}(s_1-1)}a_1+24\Sigma(-1)^{\frac{1}{2}(s_2-1)}a_2+8\Sigma(-1)^{\frac{1}{2}(s_3-1)}a_3\\
+24\Sigma(-1)^{\frac{1}{2}(s_4-1)}a_4+12\Sigma(-1)^{\frac{1}{2}(s_5-1)}a_5\\
+\ 6\Sigma(-1)^{\frac{1}{2}(s_6-1)}a_6+\Sigma(-1)^{\frac{1}{2}(s_7-1)}a_7,
\end{aligned}$$

where the Σ's relate to all the partitions of each form, and s_1 is the sum of the elements in the partition 1°, s_2 in the partition 2°, &c.

§ 58. Consider now the sum of the primary derivates of m when m is of the form $4k+1$. By adding the derivates in which the sign of a is positive in 1° (§ 51), and multiplying by $(-1)^{\frac{1}{2}(s_1-1)}$, we obtain $48(-1)^{\frac{1}{2}(s_1-1)}a_1$ as the sum. The corresponding sum in the case of 2° is derivable from this result by division by 2 (§§ 53, 55) and is therefore $24(-1)^{\frac{1}{2}(s_2-1)}a_2$; the corresponding sum for 3° is derivable from 1° by division by 6, and is therefore $8(-1)^{\frac{1}{2}(s_3-1)}a_3$; and those for 4°, 5°, 6°, 7° are derivable by division by 2, 4, 8, 48, and are therefore $24(-1)^{\frac{1}{2}(s_4-1)}a_4$, $12(-1)^{\frac{1}{2}(s_5-1)}a_5$, $6(-1)^{\frac{1}{2}(s_6-1)}a_6$, and $(-1)^{\frac{1}{2}(s_7-1)}a_7$.

These are the same quantities as those which, subject to the sign Σ, occur in the expression for $\lambda(m)$, and we therefore find that, when m is of the form $4k+1$,

$\lambda(m)$ = the sum of the primary derivates of all the partitions of m into four squares.

It is evident that this result also holds good when m is of the form $4k+3$, for then $\lambda(m)=0$ (§ 27); and the sum of the primary derivates in 1° is zero, and therefore also in $2^\circ, 3^\circ, \ldots, 7^\circ$.

Value of $P(m)$ in terms of primary derivates, § 59.

§ 59. Consider now the sum of the squares of the primary derivates of m. In this case we do not have to distinguish between the cases of $m = 4k+1$ and $m = 4k+3$, and we only require the sum of the squares of the derivates written down in 1° (§ 51), as the factor $(-1)^{\frac{1}{2}(s-1)}$ disappears when squared.

The sum of the three squares

$$(a+ib+c+id)^2, \quad (a+ib+ic+d)^2, \quad (a+b+ic+id)^2$$

is

$$3a^2-b^2-c^2-d^2+(4i+2)\,a\,(b+c+d)+(4i-2)\,(bc+bd+cd),$$

and by deriving from this expression seven others by changing the signs of b, of c, of d, of b and c, of b and d, of c and d, and of b, c, d, and adding the eight expressions, and doubling, we obtain for the sum of the squares of the primary derivates in 1° the value

$$16\,(3a_1^2-b_1^2-c_1^2-d_1^2).$$

Thus the sum of the squares of the primary derivates of all the partitions of m is

$$\Sigma\{16\,(3a_1^2-b_1^2-c_1^2-d_1^2)+8\,(3a_2^2-b_2^2-2c_2^2)+8\,(a_3^2-b_3^2)$$
$$+8\,(3a_4^2-b_4^2-c_4^2)+4\,(3a_5^2-2b_5^2)+2\,(3a_6^2-b_6^2)+a_7^2\},$$

the second and succeeding terms being derived from the first (§§ 53, 55).

This expression is equal to $P(m)$, for the value of $P(m)$ is

$$\tfrac{1}{24}\Sigma^{(r)}\,(3a^2-b^2-c^2-d^2) \quad (\S\,27),$$

which, when expanded,

$$=\tfrac{1}{24}\Sigma\{384\,(3a_1^2-b_1^2-c_1^2-d_1^2)+192\,(3a_2^2-b_2^2-2c_2^2)+\ldots\}.$$

Thus we find that $P(m)$ is equal to the sum of the squares of the primary derivates of the partitions of m.

Value of $Q(m)$ in terms of primary derivates, §§ 60–61.

§ 60. In the case of the cubes it is necessary to treat separately the cases of $m = 4k+1$ and $4k+3$.

First, suppose m of the form $4k+1$.

The sum of the three cubes

$$(a+ib+c+id)^3, \quad (a+ib+ic+d)^3, \quad (a+b+ic+id)^3$$

may be expressed in the form

$$3a^3-(2i-4)(b^3+c^3+d^3)+3(2i+1)a^2(b+c+d)-3a(b^2+c^2+d^2)$$
$$-3(b^3+c^2+d^2)(b+c+d)+6(2i-1)a(bc+bd+cd)-18bcd;$$

and, changing the signs of b, of c, of d, of b and c, ..., and adding the eight expressions, and doubling, we obtain as result

$$48a^3-48a(b^2+c^2+d^2).$$

Thus the sum of the cubes of the primary derivates of 1° is $48(-1)^{\frac{1}{2}(s_1-1)}\{a_1^3-a_1(b_1^2+c_1^2+d_1^2)\}$, and therefore the sum of the cubes of the primary derivates in 2°, 3°, ... are

$$24(-1)^{\frac{1}{2}(s_2-1)}\{a_2^3-a_2(b_2^2+2c_2^2)\},\ 8(-1)^{\frac{1}{2}(s_3-1)}(a_3^3-3a_3b_3^2),\ \ldots.$$

Now, when m is of the form $4k+1$,

$$Q(m)=\tfrac{1}{8}\Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)}a(a^2-b^2-c^2-d^2) \qquad (\S\,27),$$

so that for this form of m, $Q(m)$ is equal to the sum of the cubes of the primary derivates of the partitions of m.

§ 61. To calculate the sum of the cubes of the primary derivates when m is of the form $4k+3$ it is convenient to use the theorem

$$(a+b+c+d)^3-(a-b+c+d)^3-(a+b-c+d)^3-(a+b+c-d)^3$$
$$+(a-b-c+d)^3+(a-b+c-d)^3+(a+b-c-d)^3-(a-b-c-d)^3=48bcd.$$

Putting for b, c, d in this identity ib, c, id; ib, ic, d; b, ic, id respectively and adding and doubling we see that the sum of the cubes of the primary derivates in 1° is

$$(-1)^{\frac{1}{2}(s_1-1)}6\times 48\times -b_1c_1d_1,$$

from which the corresponding results for 2°, 3°, ... may be derived as before.

Now, when m is of the form $4k+3$,

$$Q(m)=\tfrac{1}{4}\Sigma^{(r)}(-1)^{\frac{1}{2}(s-1)}bcd\ (\S\,27),$$

which $\quad =96\Sigma(-1)^{\frac{1}{2}(s_1-1)}b_1c_1d_1+48\Sigma(-1)^{\frac{1}{2}(s_2-1)}b_2c_2d_2+\ldots.$

It follows, therefore, that, in the case of $m=4k+3$, the sum of the cubes of the primary derivates of m is equal to $-3Q(m)$.

Value of $\Omega(m)$ in terms of primary derivates, § 62.

§ 62. Treating the fourth powers in the same manner as the squares in § 59 we find that, for both forms of m, the sum

of the fourth powers of the primary derivates in 1° is equal to

$$48(a_1^4+b_1^4+c_1^4+d_1^4-2a_1^2b_1^2-2a_1^2c_1^2-2a_1^2d_1^2-2b_1^2c_1^2-2b_1^2d_1^2-2c_1^2d_1^2)$$

and comparing this expression with the second value of $\Omega(m)$ in §27 we see that $\Omega(m)$ is equal the sum of the fourth powers of the primary derivates of the partitions of m.

The functions $\lambda(m)$, $P(m)$, $Q(m)$, $\Omega(m)$, §§63–66.

§63. If we introduce a new function $Q'(m)$ derived from $Q(m)$ by the relations

$$\begin{aligned} Q'(m) &= \quad Q(m), \text{ if } m \text{ is of the form } 4k+1 \\ &= -3\,Q(m), \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 4k+3, \end{aligned}$$

then the four functions $\lambda(m)$, $P(m)$, $Q'(m)$, $\Omega(m)$ are respectively equal to the sum of the first, second, third, and fourth powers of the primary derivates of the partitions of m into four squares.

All these four functions satisfy the relation

$$\phi(m_1m_2)=\phi(m_1)\,\phi(m_2)$$

if m_1 and m_2 are prime to each other.

§64. This relation was only imperfectly satisfied in the case of the function $Q(m)$, viz. if one or both of m_1 and m_2 are of the form $4k+1$, then $Q(m_1m_2)=Q(m_1)\,Q(m_2)$, but if both are of the form $4k+3$ the relation is

$$Q(m_1m_2)=-3\,Q(m_1)\,Q(m_2).*$$

This irregularity disappears when $Q(m)$ is replaced by $Q'(m)$.

§65. It may be remarked that the elliptic-function expansions in which $Q'(m)$ occurs as a coefficient are

$$\begin{aligned} k^{\frac{1}{2}}k'^2(1+6k+k^2)\,\rho^5 &= 2\Sigma_1^\infty Q'(m)\,q^{\frac{1}{4}m}, \\ kk'(1+k^2)\,\rho^5 &= 4\Sigma_1^\infty Q'(m)\,q^{\frac{1}{2}m}, \\ k^2k'^{\frac{1}{2}}(1+k'^2)\,\rho^5 &= 32\Sigma_1^\infty Q'(m)\,q^m.\dagger \end{aligned}$$

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* Vol. XXXVII., p. 47.

† The corresponding expansions involving $Q(m)$ are

$$\begin{aligned} k^{\frac{1}{2}}k'^2(1-k)^2\rho^5 &= 2\Sigma_1^\infty Q(m)\,q^{\frac{1}{4}m}, \\ kk'^3\rho^5 &= 4\Sigma_1^\infty Q(m)\,q^{\frac{1}{2}m}, \\ k^2k'^{\frac{3}{2}}\rho^5 &= 16\Sigma_1^\infty Q(m)\,q^m \end{aligned}$$

(vol. XXXVI., pp. 349–350).

§ 66. It is only in connection with $\lambda(m)$ and $Q(m)$ that it is necessary to introduce primary derivates, as the functions $P(m)$ and $\Omega(m)$ are equal to one half the sum of the squares, and of the fourth powers, of *all* the derivates.

The primary derivates, §§ 67–68.

§ 67. The primary derivates possess a general resemblance to the primary numbers having m as norm which have been used in connection with the functions $\chi_r(m)$, viz. if a^2+b^2 is a partition of m into two squares, a being uneven,

$$(-1)^{\frac{1}{2}(a+b-1)}(a+ib)$$

is a primary number having m as norm, and if $a^2+b^2+c^2+d^2$ is a partition of m into four squares, a being uneven and b, c, d even, or a being even and b, c, d uneven,

$$(-1)^{\frac{1}{2}(a+b+c+d-1)}(a+ib+i^2c+i^3d)$$

is a primary derivate; but in the case of two squares the primary numbers are divisors of m and this is not the case with the primary derivates.

Although the primary derivates do not seem to be analogous to divisors or to offer much help towards a simpler explanation of the multiplication property than that given in vol. XXXVII., pp. 36–48, still I was very glad to obtain any quantities which would express $\lambda(m)$, $P(m)$, $Q'(m)$, $\Omega(m)$ as sums of consecutive powers.

§ 68. The primary derivates have been defined as derived from the representations by the formula $a+ib+i^2c+i^3d$, but they might have been derived by the simpler formula $a+b+i(c+d)$. In fact, if, from the partitions of m, we form every possible combination of the four elements (each having the positive or negative sign) by adding two and adding the sum of the other two multiplied by i, there being therefore one such complex number corresponding to every representation of m, then among these complex numbers those which are primary are the primary derivates.

Quaternions having m as norm, § 69.

§ 69. The complex number $a+ib$ is a divisor of a^2+b^2 and the functions $\chi_r(n)$ are defined arithmetically as sums of

powers of numbers having n as norm, so that it seems natural to inquire if $P(m)$, $Q(m)$, $\Omega(m)$, which are derived from representations by four squares, can be expressed by means of the quaternion divisors of m which have m as norm.

Forming the square and the fourth power of $a+ib+jc+kd$, we find

$$(a+ib+jc+kd)^2 = a^2-b^2-c^2-d^2+2a(ib+jc+kd),$$

and $$(a+ib+jc+kd)^4 = a^4+b^4+c^4+d^4$$
$$-6a^2b^2-6a^2c^2-6a^2d^2+2b^2c^2+2b^2d^2+2c^2d^2$$
$$+4a(a^2-b^2-c^2-d^2)(ib+jc+kd).$$

When a, b, c, d are different and non-zero there are 384 quaternions involving these four letters and having $a^2+b^2+c^2+d^2$ as norm,* and it is easy to see that the sum of the fourth powers of these 384 quaternions is equal to

$$384(a^4+b^4+c^4+d^4-2a^2b^2-2a^2c^2-2a^2d^2-2b^2c^2-2b^2d^2-2c^2d^2),$$

whence it follows that $\Omega(n)$ is equal to one-eighth of the sum of the fourth powers of all the quaternions having m as norm.

As regards the squares, however, I do not see how to connect $P(m)$ with the sum of the squares of quaternions having m as norm.

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* A quaternion $a+ib+jc+kd$ gives rise to 23 other quaternions by permuting the letters a, b, c, d, and a quaternion with all its signs positive gives rise to fifteen others by merely changing the signs of the terms, so that the total number $=24\times16$.

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ON COLLINEATIONS OF SPACE.

By C. M. Jessop.

IT has been remarked, without formal proof, by Segre*, in his paper on the classification of homographies, that the collineation determined by the equations

$$\rho x_i = \sum_1^4 a_{ik} y_k, \quad (i = 1, 2, 3, 4)$$

may be obtained by a double reciprocity, viz. by taking the poles x and y of any plane with regard to the quadrics U and V; and that the invariants of the collineation are expressible in terms of the invariants of the system $U + \lambda V$.

The present paper begins by showing that any collineation is equivalent to a double reciprocity with regard to two corresponding members of the triply infinite system of quadrics $\sum_1^4 \lambda_i S_i = 0$; the dependence of the invariants of the collineation upon those of a pair of quadrics is discussed, and hence the geometrical meaning of their vanishing is easily determined; other problems connected with collineation are investigated by the method described.

§ 1. The equations giving the general space-collineation are

$$\rho x_i = \sum_1^4 a_{ik} y_k, \quad i = 1, 2, 3, 4.$$

Any collineation has at least one 'united' or self-corresponding point, arising from the quartic for ρ obtained by making $x_i \equiv y_i$ in these equations; and also at least one 'united' plane. Take this point as the vertex A_1 of the tetrahedron of reference, and this plane as the plane α_4 of the tetrahedron. In α_4 there is at least one line which coincides with its corresponding line (and passing through A_1); take this line as the edge A_1A_2; while the intersection of the corresponding planes

$$x_4 = 0, \quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = 0,$$

may be taken as the edge A_2A_4 of the tetrahedron of reference, whence $a_{13} = a_{14} = 0$. The plane α_2 is now fixed; to it there

* *Sulla teoria e sulla classificazione delle omografie*, Atti Lincei (3), XIX. (1884).

corresponds a definite plane, and we may take the point in which the latter meets A_3A_4 as the vertex A_4.

The effect of the choice of tetrahedron of reference which has been made, is to reduce the equations of the collineation to the following:

$$\begin{aligned}
\rho x_1 &= a_{11}y_1 + a_{12}y_2, \\
\rho x_2 &= \quad a_{22}y_2 + a_{23}y_3, \\
\rho x_3 &= \quad\quad a_{33}y_3 + a_{34}y_4, \\
\rho x_4 &= \quad\quad\quad a_{44}y_4.
\end{aligned}$$

This form includes both the most general collineation possessing four distinct united points, and also every special collineation, up to that which has only one united point.

Denoting by U the quadratic form $\Sigma\alpha_{ik}x_ix_k\,(\alpha_{ik}=\alpha_{ki})$; multiply these equations respectively by α_{11}, α_{12}, α_{13}, α_{14} and add, we obtain

$$\rho\frac{\partial U}{\partial x_1} = a_{11}\alpha_{11}y_1 + (a_{12}\alpha_{11} + a_{22}\alpha_{12})\,y_2 + (a_{23}\alpha_{12} + a_{33}\alpha_{13})\,y_3 + (a_{34}\alpha_{13} + a_{44}\alpha_{14})\,y_4,$$

similarly

$$\rho\frac{\partial U}{\partial x_2} = a_{11}\alpha_{12}y_1 + (a_{12}\alpha_{12} + a_{22}\alpha_{22})\,y_2 + (a_{23}\alpha_{22} + a_{33}\alpha_{23})\,y_3 + (a_{34}\alpha_{23} + a_{44}\alpha_{42})\,y_4,$$

$$\rho\frac{\partial U}{\partial x_3} = a_{11}\alpha_{13}y_1 + (a_{12}\alpha_{13} + a_{22}\alpha_{23})\,y_2 + (a_{23}\alpha_{23} + a_{33}\alpha_{33})\,y_3 + (a_{34}\alpha_{33} + a_{44}\alpha_{43})\,y_4,$$

$$\rho\frac{\partial U}{\partial x_4} = a_{11}\alpha_{14}y_1 + (a_{12}\alpha_{14} + a_{22}\alpha_{24})\,y_2 + (a_{23}\alpha_{24} + a_{33}\alpha_{34})\,y_3 + (a_{34}\alpha_{34} + a_{44}\alpha_{44})\,y_4.$$

If now the expressions on the right of these equations are the respective partial derivatives with regard to y of a quadratic function V, the following conditions must hold:

$$\begin{aligned}
&\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22} = a_{11}\alpha_{12}, \quad && \alpha_{22}\alpha_{23} + \alpha_{23}\alpha_{33} = \alpha_{13}\alpha_{12} + \alpha_{23}\alpha_{22}, \\
&\alpha_{12}\alpha_{23} + \alpha_{13}\alpha_{33} = a_{11}\alpha_{13}, && \alpha_{23}\alpha_{34} + \alpha_{24}\alpha_{44} = \alpha_{14}\alpha_{12} + \alpha_{24}\alpha_{22}, \\
&\alpha_{13}\alpha_{34} + \alpha_{14}\alpha_{44} = a_{11}\alpha_{14}, && \alpha_{33}\alpha_{34} + \alpha_{34}\alpha_{44} = \alpha_{24}\alpha_{23} + \alpha_{34}\alpha_{33}.
\end{aligned}$$

These six equations are seen to be independent of each other; hence the 10 quantities α_{ik}, satisfying the above equations, are definite linear functions of four arbitrary quantities; and it is proved that any collineation whatever can be reduced to the form

$$\rho \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial y_i}, \quad i=1, 2, 3, 4 \ldots\ldots\ldots\ldots\ldots\text{(I)},$$

where U is of the form $\lambda S_1 + \mu S_2 + \nu S_3 + \sigma S_4$,

V „ „ $\lambda S_1' + \mu S_2' + \nu S_3' + \sigma S_4'$,

and the quadrics $S_1' \ldots S_4'$ are determined by the collineation.

The equations I. state that *corresponding points x_i and y_i in any collineation are the poles of the various planes u_i of space with regard to two quadrics U and V;* It follows similarly that *corresponding planes u_i and v_i of the collineation are the polar planes of the points of space with regard to U and V.*

In the general case when U and V have a common self-conjugate tetrahedron, the vertices of this tetrahedron are the united points, and the faces the united planes of the collineation. Here the triply infinite system of quadrics have a common self-conjugate tetrahedron.

Classification of collineations.

§ 2. There are 5 principal types of pairs of equations for U and V corresponding to special relationships between them.

The *general* type is

$$U = \Sigma x_i^2, \quad V = \Sigma a_i x_i^2,$$

the corresponding *general* collineation is then

$$\rho x_i = a_i y_i, \quad i=1, 2, 3, 4.$$

In the sub-case [11 (11)] $a_3 = a_4$, the quadrics touch in two points and intersect in 2 conics, the collineation is *axial*, since we have $x_3/x_4 = y_3/y_4$; the line joining corresponding points meets the edge A_1A_2 of the tetrahedron of reference.

In [(11) (11)], the quadrics have 4 common generators forming a quadrilateral, $a_1 = a_2$, $a_3 = a_4$ and we have, in the corresponding collineation,

$$\frac{x_1}{x_2} = \frac{y_1}{y_2}, \quad \frac{x_3}{x_4} = \frac{y_3}{y_4};$$

the collination is *doubly axial*, the line joining 2 corresponding points meets two fixed lines.

In [1 (111)] $a_2 = a_3 = a_4$, the quadrics touch along a conic and

$$x_2 : x_3 : x_4 = y_2 : y_3 : y_4 ;$$

the collineation is here *central*, the join of corresponding points passes through the fixed point A_1 while $\{APLQ\} = a$, L being the point in which APQ meets the fixed plane α_1.

In [112] we have

$$U \equiv x_1^2 + x_2^2 + 2x_3x_4,$$
$$V \equiv a_{11}x_1^2 + a_{22}x_2^2 + 2a_{34}x_3x_4 + x_3^2 ;$$

here U and V touch; the corresponding collineation is

$$\rho x_1 = a_{11}y_1, \ \rho x_2 = a_{22}y_2, \ \rho x_3 = a_{34}y_3, \ \rho x_4 = a_{34}y_4 + y_3 .$$

In [13] we have

$$U \equiv x_1^2 + x_3^2 + 2x_2x_4,$$
$$V \equiv a_{11}x_1^2 + a_{33}(x_3^2 + 2x_2x_4) + 2x_2x_3 ;$$

here U and V have stationary contact; the collineation is

$$\rho x_1 = a_{11}y_1, \ \rho x_2 = a_{33}y_2, \ \rho x_3 = a_{33}y_3 + y_2, \ \rho x_4 = a_{33}y_4 + y_3 .$$

In [22] we have

$$U \equiv x_1x_2 + x_3x_4,$$
$$V \equiv 2a_{12}x_1x_2 + 2a_{34}x_3x_4 + x_1^2 + x_3^2 ;$$

here U and V touch along a generator; the collineation is

$$\rho x_1 = a_{12}y_1, \ \rho x_2 = a_{12}y_2 + y_1, \ \rho x_3 = a_{34}y_3, \ \rho x_4 = a_{34}y_4 + y_3 .$$

In [4] we have

$$U \equiv x_1x_4 + x_2x_3,$$
$$V \equiv 2a_{14}(x_1x_4 + x_2x_3) + 2x_1x_3 + x_2^2 ;$$

the quadrics have a common generator which touches their residual cubic of intersection; the collineation is

$$\rho x_1 = a_{14}y_1, \ \rho x_2 = a_{14}y_2 + y_1, \ \rho x_3 = a_{14}y_3 + y_2, \ \rho x_4 = a_{14}y_4 + y_3 .$$

Hence the varieties of collineation as deduced from the different relationships between U and V are seen to coincide with the known varieties of space-collineation.*

The invariants of the collineation.

§ 3. By aid of this view of space-collineations we can find the geometrical meaning of the vanishing of the 4 invariants.

* See Segre, *loc. cit.*

In the collineation $\rho x_i = \Sigma a_{ik} y_k$, the equation to determine the united points is $|a_{ik} - \rho| = 0$, or

$$\rho^4 - \rho^3 \iota_1 + \rho^2 \iota_2 - \rho^3 \iota_3 + |a_{ik}| = 0.$$

The meaning of the vanishing of the invariants ι_1, ι_2, ι_3, $|a_{ik}|$ is easily determined by the method of this paper. For the collineation being $\rho(\partial U/\partial x_i) = \partial V/\partial y_i$, the equation for the determination of ρ is seen to be

$$\rho^4 \Delta - \rho^3 \Theta + \rho^2 \Phi - \rho \Theta' + \Delta' = 0,$$

where Δ, Θ, Φ, Θ', Δ' have their usual meanings.* Hence

$$\iota_1 = \frac{\Theta}{\Delta}, \quad \iota_2 = \frac{\Phi}{\Delta}, \quad \iota_3 = \frac{\Theta'}{\Delta}, \quad \iota_4 = \frac{\Delta'}{\Delta}.$$

If $\Delta' = 0$, V is a cone, and the polar plane of y for V passes through the vertex A of V; hence the corresponding point x lies in a fixed plane, viz. the polar plane α of A for U. Again to any point x of space there corresponds in general no definite point y, since the polar plane of x for U does not in general pass through A; while, if x lies in α, its polar plane for U is some plane u passing through A, and any point y in the line r, polar to the plane u for V, corresponds to x.

Again, if $\Theta = 0$, we can inscribe in V a tetrahedron self-conjugate with regard to U. This condition being supposed satisfied, take 4 such points Y_i on V; their polar planes $u_1 \ldots u_4$ for V are the tangent planes to V at these points, and since u_4 passes through Y_4 its pole X_4 for U lies upon the polar plane of Y_4 for U, *i.e.* the plane $Y_1 Y_2 Y_3$, *i.e.* X_4 lies upon $Y_1 Y_2 Y_3$, and so on. Hence $\iota_1 = 0$ *expresses the condition that the tetrahedron* $X_1 X_2 X_3 X_4$, *corresponding to the tetrahedron* $Y_1 Y_2 Y_3 Y_4$, *is inscribed in the latter.*

Similarly $\iota_3 = 0$, or $\Theta' = 0$, expresses the condition that a tetrahedron of points X_i can be found *circumscribed* to the tetrahedron of corresponding points Y_i.

If $\Phi = 0$, then the edges of a tetrahedron self-conjugate with regard to either quadric will touch the other†. Refer, therefore, the two quadrics to such a tetrahedron, and their equations will be

$$\Sigma a_i^2 x_i^2 - 2\Sigma a_i a_k x_i x_k = 0, \quad \Sigma x_i^2 = 0;$$

the equations of the collineation will then be

$$\rho y_1 = a_1(\ a_1 x_1 - a_2 x_2 - a_3 x_3 - a_4 x_4),$$
$$\rho y_2 = a_2(-a_1 x_1 + a_2 x_2 - a_3 x_3 - a_4 x_4), \text{ \&c.}$$

* See Salmon, *Geometry of three Dimensions*, chap. IX., 4th ed.
† Salmon, *Geometry of Three Dimensions*, p. 175.

Hence to the edge A_1A_2 of the tetrahedron of reference will correspond the line

$$-a_1x_1-a_2x_2+a_3x_3-a_4x_4=0$$
$$-a_1x_1-a_2x_2-a_3x_3+a_4x_4=0,$$

or
$$a_1x_1+a_2x_2=a_3x_3-a_4x_4=0;$$

but this is a line meeting A_1A_2 and A_3A_4; and so on. Hence *two tetrahedra of corresponding points can be found such that each pair of opposite edges of the one meet the corresponding pair of opposite edges of the other.**

Conjugate tetrahedra.

§ 4. If the polar planes $\alpha_1'\ldots\alpha_4'$ of any four points $A_1A_2A_3A_4$ (not co-planar) be taken with regard to any quadric F, we obtain a new tetrahedron whose vertices are the poles of the faces of the original tetrahedron with respect to F. Such a pair of tetrahedra may be called *conjugate*.

It can be shown that the lines of intersection $\alpha_1\alpha_1'$, $\alpha_2\alpha_2'$, $\alpha_3\alpha_3'$, $\alpha_4\alpha_4'$ of corresponding faces *lie on the same regulus*. For taking the points A_i as vertices of the tetrahedron of reference, and taking F to be

$$\Sigma a_{ii}x_i^2+2a_{12}(x_1x_2+x_3x_4)+2a_{13}(x_1x_3+x_2x_4)$$
$$+2a_{14}(x_1x_4+x_2x_3)=0,$$

consider the quadric F' whose equation is

$$\Sigma x_i^2+\frac{a_{13}^2+a_{14}^2}{a_{13}a_{14}}(x_1x_2+x_3x_4)+\frac{a_{14}^2+a_{12}^2}{a_{12}a_{14}}(x_1x_3+x_2x_4)$$
$$+\frac{a_{12}^2+a_{13}^2}{a_{12}a_{13}}(x_1x_4+x_2x_3)=0.$$

The equation of the latter quadric may be written in the form

$$x_1\left(x_1+\frac{a_{13}^2+a_{14}^2}{a_{13}a_{14}}x_2+\frac{a_{14}^2+a_{12}^2}{a_{12}a_{14}}x_3+\frac{a_{12}^2+a_{13}^2}{a_{12}a_{13}}x_4\right)$$
$$+(a_{12}x_2+a_{13}x_3+a_{14}x_4)\left(\frac{x_2}{a_{12}}+\frac{x_3}{a_{13}}+\frac{x_4}{a_{14}}\right)=0,$$

which meets the plane α_1 (of reference) in the lines

$$x_1=0,\quad a_{12}x_2+a_{13}x_3+a_{14}x_4=0,$$
$$x_1=0,\qquad \frac{x_2}{a_{12}}+\frac{x_3}{a_{13}}+\frac{x_4}{a_{14}}=0.$$

* This result was shown otherwise by Segre, *Mem. Acc. Torino* II., 37.

The former of these lines is the intersection of α_1 with the plane $a_{11}x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4=0$, *i.e* the plane α_1'. Similarly the lines $\alpha_2\alpha_2'$, $\alpha_3\alpha_3'$, $\alpha_4\alpha_4'$ are seen to lie upon F'.

The converse of this result is also true, since if we take the tetrahedron Δ as that of reference, then since any quadric F' inscribed in the tetrahedron of reference, has an equation of the form

$$F' \equiv \Sigma x_i^2 + \frac{a_{12}^2+a_{13}^2}{a_{12}a_{13}}(x_1x_4+x_2x_3) + \frac{a_{13}^2+a_{14}^2}{a_{15}a_{14}}(x_1x_2+x_3x_4) + \frac{a_{14}^2+a_{12}^2}{a_{12}a_{14}}(x_1x_3+x_2x_4) = 0,$$

and since a side α' of the tetrahedron Δ', by hypothesis, meets the corresponding side α of Δ in one of the lines of contact of F' with α, it follows that Δ' is formed either by the planes u_1 or by the planes u_1', where

$$u_1 \equiv \lambda_1x_1+a_{12}x_2+a_{13}x_3+a_{14}x_4=0,$$

$$u_1' \equiv \lambda_1x_1+\frac{x_2}{a_{12}}+\frac{x_3}{a_{13}}+\frac{x_4}{a_{14}}=0, \text{ \&c.}$$

One of these tetrahedra is conjugate to Δ for the quadric whose equation is $\Sigma\lambda_ix_i^2+\Sigma a_{ik}x_ix_k=0$, the other is conjugate to Δ for the quadric whose equation is

$$\Sigma\lambda_ix_i^2+\Sigma\frac{x_ix_k}{a_{ik}}=0.$$

Hence *if* 4 *sides of a tetrahedron meet* 4 *sides of another tetrahedron in* 4 *lines belong to one regulus of a given quadric, the tetrahedra are conjugate with respect to* ∞^4 *quadrics* (since the quantities λ_i are arbitrary).

The collineation of period 4.

§ 5. Among collineations, one of special interest is the one which cyclically permutes the point of space in groups of four; by taking as tetrahedron of reference such a group it is seen that the equations giving the collineation are of the form

$$\rho x_1=\lambda_1y_2,\ \rho x_2=\lambda_2y_3,\ \rho x_3=\lambda_3y_4,\ \rho x_4=\lambda_4y_1.$$

The united points of the collineation are determined, on making $x_i\equiv y_i$, from the equation $\rho^4=\lambda_1\lambda_2\lambda_3\lambda_4$; and thus two varieties of this collineation arise, viz. when $\lambda_1\lambda_2\lambda_3\lambda_4$ is *positive* and *negative* respectively; in the former case two united points are real and two imaginary; in the latter they are all imaginary.

By suitable choice of the coordinate system the collineation is seen, therefore, to be given either by

$$\rho x_1 = y_2,\ \rho x_2 = y_3,\ \rho x_3 = y_4,\ \rho x_4 = y_1;$$

or by $\rho x_1 = -y_2,\ \rho x_2 = y_3,\ \rho x_3 = y_4,\ \rho x_4 = y_1.$

Consider the first of these collineations; it is obtained from the equations

$$\rho \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial y_i}, \quad i = 1, 2, 3, 4$$

if we take $U \equiv x_2^2 + x_4^2 + 2x_1x_3,\ V \equiv 2(y_1y_4 + y_2y_3),$

or if we take $U \equiv x_1^2 + x_3^2 + 2x_2x_4,\ V \equiv 2(y_1y_2 + y_3y_4).$

Since the collineation has four distinct united points, the preceding four quadrics have a common self-conjugate tetrahedron.

Consider the ∞^3 quadric Φ_1 having the equation

$$u_1(x_1^2 + x_3^2 + 2x_2x_4) + 2u_2(x_1x_2 + x_3x_4) + u_3(x_2^2 + x_4^2 + 2x_1x_3) + 2u_4(x_1x_4 + x_2x_3) = 0,$$

or $u_1S_1 + u_2S_2 + u_3S_3 + u_4S_4 = 0$, for which the tetrahedron of united points is self-conjugate; the polar planes of the vertices $A_1 \ldots A_4$ of the tetrahedron of reference are

$$\alpha_1' \equiv u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0,$$
$$\alpha_2' \equiv u_2x_1 + u_3x_2 + u_4x_3 + u_1x_4 = 0,$$
$$\alpha_3' \equiv u_3x_1 + u_4x_2 + u_1x_3 + u_2x_4 = 0,$$
$$\alpha_4' \equiv u_4x_1 + u_1x_2 + u_2x_3 + u_3x_4 = 0;$$

each plane arising out of the preceding one by cyclical permutations of the u_i. Now α_1', α_4', α_3', α_2' are the planes $u_x = 0$, and those into which this plane is successively permutated by the collineation; hence, *having given one tetrad of points A_i determined by the collineation, the others are obtained as vertices of tetrahedra conjugate to the first tetrahedron for different members of the triply infinite system of quadrics* $\Sigma' u_i S_i = 0$.

Moreover, taking instead of Φ_1 the quadrics

$$\Phi_2 \equiv u_4S_1 + u_1S_2 + u_2S_3 + u_3S_4 = 0,$$
$$\Phi_3 \equiv u_3S_1 + u_4S_2 + u_1S_3 + u_2S_4 = 0,$$
$$\Phi_4 \equiv u_2S_1 + u_3S_2 + u_4S_3 + u_1S_4 = 0,$$

which are formed from Φ_1 by cyclical permutation of the u_i, it is obvious that the polar planes of $A_1 \ldots A_4$, forming the

tetrahedron Δ', for these quadrics, give the *same* conjugate tetrahedron Δ' as before; hence, since Δ and Δ' are conjugate in four different ways,* the intersections of their sides are co-regular in a four-fold manner. It may be noticed that the lines belonging to the same regulus are in these 4 cases, respectively,

$\alpha_1\alpha_1'$, $\alpha_2\alpha_2'$, $\alpha_3\alpha_3'$, $\alpha_4\alpha_4'$, since Δ and Δ' are conjugate for Φ_1,
$\alpha_1\alpha_4'$, $\alpha_2\alpha_1'$, $\alpha_3\alpha_2'$, $\alpha_4\alpha_3'$, ,, ,, ,, Φ_2,
$\alpha_1\alpha_3'$, $\alpha_2\alpha_4'$, $\alpha_3\alpha_1'$, $\alpha_4\alpha_2'$, ,, ,, ,, Φ_3,
$\alpha_1\alpha_2'$, $\alpha_2\alpha_3'$, $\alpha_3\alpha_4'$, $\alpha_4\alpha_1'$, ,, ,, ,, Φ_4.

We may also obtain these results as follows:—let Q, R, S be the positions into which P is successively displaced by the given collineation, U and V being a pair of quadrics determining the collineation, and let

$Q'R'S'$ be the polar plane of P for U,
$R'S'P'$,, ,, ,, Q ,, ,,
$S'P'Q'$,, ,, ,, R ,, ,,
$P'Q'R'$,, ,, ,, S ,, ,,

then P' is the pole of the plane QRS for U, and $P'Q'R'S'$ is conjugate to $PQRS$ for U; but since Q corresponds to P in the collineation

the pole of $Q'R'S'$ for V is Q,
and similarly ,, ,, $R'S'P'$,, R,
,, ,, $S'P'Q'$,, S,
,, ,, $P'Q'R'$,, P;

hence the pole of QRS for U being P' and for V being S' it follows that P' and S' are corresponding points in the collineation; similarly S' and R', R' and Q', Q' and P' are corresponding points. By taking other quadrics of the triple infinite set to which U belongs we should obtain all the tetrahedra $P'Q'R'S'$ of closed set of four points. This proof applies to either of the two collineations of period four.

It is also to be observed that the tetrahedra $PQRS$, $P'Q'R'S'$ are conjugate both for U and for V; hence *the necessary and sufficient condition that two quadrics should*

* Schur, *Math. Ann.* XX., discusses, from another point of view, such a relationship of two tetrahedra, *Ueber ein besondre Classe von Flächen vierter Ordnung.*

possess a common pair of conjugate tetrahedra is that $\Theta = \Phi = \Theta' = 0$; since the collineation being

$$\rho x_1 = \lambda_1 y_2,\ \rho x_2 = \lambda_2 y_3,\ \rho x_3 = \lambda_3 y_4,\ \rho x_4 = \lambda_4 y_1,$$

the equation to determine the united points is

$$\rho^4 = \lambda_1\lambda_2\lambda_3\lambda_4,$$

whence $$i_1 = i_2 = i_3 = 0.$$

The collineation of period three.

§ 6. The collineation established by the equation

$$\rho x_1 = y_1,\ \rho x_2 = \lambda_2 y_3,\ \rho x_3 = \lambda_3 y_4,\ \rho x_4 = \lambda_4 y_2,$$

where $\lambda_2\lambda_3\lambda_4 = 1$, is of period three; and it is easy to see that any collineation whose period is three can be defined by equations of this form. It possesses a line of united points, viz.

$$x_2 = \lambda_2 x_3,\quad x_3 = \lambda_3 x_4;$$

while there are an infinite number of planes which correspond to themselves, forming a pencil whose *axis* is

$$x_1 = 0,\quad x_2 + \lambda_2 x_3 + \lambda_2\lambda_3 x_4 = 0.$$

The quadrics U and V which give rise to the collineation here form, exceptionally, a quadruply infinite set, viz.

$$\lambda x_1^2 + 2\mu\,(x_1x_2 + x_1x_3 + x_1x_4) + \nu\,(x_2^2 + 2x_3x_4) + \rho\,(x_3^2 + 2x_2x_4) + \sigma\,(x_4^2 + 2x_2x_3) = 0.$$

We now consider the necessary conditions in order that two tetrahedra $A_1A_2A_3A_4$, $A_1'A_2'A_3'A_4'$ may be unaltered, both by a collineation of period four and by a collineation of period three. It is clear that, in order that this may happen, two vertices, say A_1 and A_1', must lie on the line of united points of the latter collineation, while the planes α_1, α_1' opposite to A_1 and A_1' must meet on the axis of this collineation.

Taking the tetrahedron $A_1A_2A_3A_4$ as that of reference, and the collineations defined by

$$\rho x_1 = y_2,\ \rho x_2 = y_3,\ \rho x_3 = y_4,\ \rho x_4 = y_1,$$

$$\sigma x_1 = y_1,\ \sigma x_2 = \lambda_2 y_3,\ \sigma x_3 = \lambda_3 y_4,\ \sigma x_4 = \lambda_4 y_2;$$

the points A_1', A_2', A_3', A_4' will be, respectively,

$$(x_1x_2x_3x_4),\ (x_4x_1x_2x_3),\ (x_3x_4x_1x_2),\ (x_2x_3x_4x_1),$$

since they are subject to the first collineation; while, since A_1' lies on the *united line* of the second collineation

$$x_2=\lambda_2 x_3, \quad x_3=\lambda_3 x_4;$$

moreover A_2', A_3', A_4' pass into each other cyclically, since they are subject to the second collineation, hence, since A_3' corresponds to A_2',

$$\sigma x_4=x_3, \ \sigma x_1=\lambda_2 x_1, \ \sigma x_2=\lambda_3 x_2, \ \sigma x_3=\lambda_4 x_4,$$

therefore $\lambda_2=\lambda_3=\pm\sqrt{\lambda_4}$, and since $\lambda_2\lambda_3\lambda_4=1$, we may take $\lambda_4=1$ (considering real collineations only), so that either $\lambda_2=\lambda_3=\lambda_4=1$, *or* $\lambda_2=\lambda_3=-\lambda_4=-1$. These conditions are easily seen to be sufficient and necessary to ensure that A_2', A_3', and A_4' are connected by the second collineation. We find, therefore, that *there is an infinite number of tetrahedra each of which is transformed into itself by the given collineation of period four, and by a collineation of period three*

$$\sigma x_1=y_1, \ \sigma x_2=y_3, \ \sigma x_3=y_4, \ \sigma x_4=y_2;$$

the faces of these tetrahedra are easily seen to be, m being arbitrary,

$$mx_1+x_2+x_3+x_4=0, \quad x_1+mx_2+x_3+x_4=0,$$
$$x_1+x_2+mx_3+x_4=0, \quad x_1+x_2+x_3+mx_4=0.$$

Similarly the faces of the tetrahedra formed by the planes

$$mx_1+x_2-x_3+x_4=0, \quad mx_4+x_1-x_2+x_3=0,$$
$$mx_3-x_1+x_2+x_4=0, \quad mx_2-x_4+x_1+x_3=0,$$

are seen to be permuted into each other by the given collineation and by the collineation

$$\sigma x_1=y_1, \ \sigma x_2=-y_3, \ \sigma x_3=-y_4, \ \sigma x_4=y_2.$$

But this is merely the modification of the previous case involved in the change to the coordinate system obtained by changing the signs of x_2 and x_4.

In the next place consider the collineation of period four represented by the equations

$$\rho x_1=-y_2, \ \rho x_2=y_3, \ \rho x_3=y_4, \ \rho x_4=y_1,$$

and let any collineation of period three be

$$\sigma x_1=y_1, \ \sigma x_2=\lambda_2 y_3, \ \sigma x_3=\lambda_3 y_4, \ \sigma x_4=\lambda_4 y_2; \ \lambda_2\lambda_3\lambda_4=1.$$

The four points A_1', A_2', A_3', A_4' will be, respectively,

$$(x_1x_2x_3x_4), \ (x_4-x_1x_2x_3), \ (x_3-x_4-x_1x_2), \ (x_2-x_3-x_4-x_1).$$

Also since A_2' is to pass into A_3', and A_3' into A_4', by the last collineation, we must have

$$\sigma x_4=x_3, \ \sigma x_1=\lambda_2 x_1, \ \sigma x_2=\lambda_3 x_2, \ \sigma x_3=-\lambda_4 x_4,$$

and also

$$\sigma' x_3 = x_2,\ \sigma' x_4 = \lambda_2 x_4,\ \sigma' x_1 = \lambda_3 x_1,\ \sigma' x_2 = -\lambda_4 x_3.$$

If x_1 were not zero we should have

$$\lambda_2 = \lambda_3 \text{ and } \lambda_2^2 = \lambda_3^2 = -\lambda_4,$$

hence with $\lambda_2\lambda_3\lambda_4 = 1$, we should have $\lambda_4^2 = -1$; excluding imaginary collineations, we must therefore have either $x_1 = 0$ and $\lambda_2 = -\lambda_3 = -\lambda_4 = 1$, *or* $x_1 = 0$, $\lambda_2 = -\lambda_3 = \lambda_4 = -1$. The point A_1' is then determined either by the equations

$$x_1 = 0,\ x_2 = x_3,\ x_3 + x_4 = 0,$$

or by the equations

$$x_1 = 0,\ x_2 + x_3 = 0,\ x_3 - x_4 = 0.$$

A corresponding result holds for the two collineations of period three which leave the vertices A_2 and A_2' unaltered, &c.; hence, *eight collineations of period three may be associated with a given collineation of period four of this species, so as to leave unaltered 2 tetrahedra belonging to the latter collineation.*

The equations of the faces of the second tetrahedron are, in the first case,

$$x_2 + x_3 - x_4 = 0,\ x_1 - x_2 + x_3 = 0,\ x_1 + x_2 + x_4 = 0,\ x_3 + x_4 - x_1 = 0.$$

The second tetrahedron is circumscribed to the first tetrahedron (that of reference), and is also easily seen to be inscribed in it.

Quartic surfaces which admit of both collineations.

§ 7. If a surface admits of the collineations

$$\rho x_1 = y_2,\ \rho x_2 = y_3,\ \rho x_3 = y_4,\ \rho x_4 = y_2;$$
$$\sigma x_1 = y_1,\ \sigma x_2 = y_3,\ \sigma x_3 = y_4,\ \sigma x_4 = y_2;$$

it will admit of all the 24 collineations obtained by associating x_1, x_2, x_3, x_4 with y_i, y_j, y_k, y_l, where i, j, k, l is any permutation of the numbers 1, 2, 3, 4. Hence such a surface is expressible in terms of the symmetric functions S_1, S_2, ... of the x_i, where $S_1 = \Sigma x_i$, $S_2 = \Sigma x_i x_j$, &c.

The quartic surfaces of this description are therefore represented by

$$AS_1^4 + BS_1^2 S_2 + CS_1 S_3 + DS_4 + ES_2^2 = 0.$$

The first four terms of the left-hand side can be expressed in the form

$$\prod_1^4 (S_1 + x_i/m) + \lambda \prod_1^4 (S_1 + x_i/n)$$

in *two* ways; for equating these expressions we have, for the determination of m, n, λ, the equations

$$1/m^2+\lambda/n^2=B/A\{1+1/m+\lambda(1+1/n)\},$$
$$1/m^3+\lambda/n^3=C/A\{1+1/m+\lambda(1+1/n)\},$$
$$1/m^4+\lambda/n^4=D/A\{1+1/m+\lambda(1+1/n)\};$$

whence $$B-C(m+n)+Dmn=0,$$

and $$B(m^2+mn+n^2+m+n)-Cmn(m+n+1)-A=1.$$

These equations determine two values for mn and $m+n$, either of which gives one way of expressing

$$AS_1^4+BS_1^2S_2+CS_1S_3+DS_4$$

in the required form. We conclude, therefore, that all quartic surfaces admitting of the stated collineations are included in the form

$$\prod_1^4(S_1+x_i/m)+\lambda\prod_1^4(S_1+x_i/n)+ES_2^2=0.$$

F. Schur* considers surfaces analogous to these, viz. surfaces of the form

$$x_1x_2x_3x_4+\alpha(x_2+x_3-x_4)(x_3+x_4-x_1)(x_1+x_2+x_4)(x_1-x_2+x_3)=0.$$

It has been shown, § 6, that these surfaces admit of a collineation of period four of the second kind, and also a collineation of period three. It is shown by Schur by synthetic methods that this surface possesses 52 lines. The determination of these lines is easy from the equation of the surface. For if we seek the conditions that the line

$$x_3-x_1=mx_4,$$
$$x_3+x_1=nx_2,$$

may lie upon the surfaces, then since

$$x_2+x_4+x_1=x_3(1/m+1/n)+x_1(1/n-1/m+1),$$
$$x_2-x_4+x_3=x_3(1/n-1/m+1)+x_1(1/n+1/m),$$

on insertion of these values we obtain, as the condition to be satisfied,

$$x_1x_3+\alpha(m+1)(n-1)[(x_1^2+x_3^2)(1/m+1/n)(1/n-1/m+1)$$
$$+x_1x_3\{(1/m+1/n)^2+(1/n-1/m+1)^2\}]\equiv 0.$$

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* *loc. cit.*, *Math. Ann.*, XX.

This requires that *either*

$$1/m+1/n=0,\ 1+\alpha(m+1)(n-1)(1/n-1/m+1)^2=0;$$

giving 4 solutions, and therefore four lines, lying on the quadric

$$\frac{x_4}{x_3-x_1}+\frac{x_2}{x_3+x_1}=0;$$

or $1/n-1/m+1=0,\ 1+\alpha(m+1)(n-1)(1/m+1/n)^2=0,$

giving 4 solutions, and therefore four lines, lying on the quadric

$$\frac{x_2}{x_3+x_1}-\frac{x_4}{x_3-x_1}+1=0.$$

Now it can easily be shown that a quadric which intersects a quartic surface in 4 lines belonging to a regulus will meet the quartic in 4 other lines belonging to the complementary regulus; hence each of the preceding quadrics will meet the quartic surface in 4 other lines. But in each case, *two* of these lines are

$$x_4=0,\ x_3-x_1=0;\ x_2=0,\ x_3+x_1=0;$$

which are the lines of intersection of two faces of the tetrahedra Δ and Δ'; excluding them, it is seen that each of the two quadrics meets the quartic in six lines.

Similarly we derive two other pairs of quadrics from considering the conditions that the lines

$$x_4-x_1=mx_3,\ x_4+x_1=nx_2;\ x_1-x_2=mx_3;\ x_1+x_2=nx_4,$$

should lie upon the surface; giving six quadrics each of which meets the surface in 6 lines; hence there arise 36 lines in all, which, with the 16 intersections of the sides of Δ and Δ' give rise to 52 lines on the given quartic surface.

Collineations which leave a quadric unaltered.

§ 8. By aid of the method of this paper, the condition discovered in another manner by Segre (*Mem. Acc. Torino* II., 37), that a collineation should leave a quadric unaltered, can easily be obtained.

For let $F\equiv\Sigma x_1^2=0$ be a quadric unaltered by the collineation determined by U and V; this requires that if u is the polar plane with regard to U of any point x on F, then the pole of u with regard to V is y, a point on F; in other words the polar reciprocal of F with regard to U is identical with the polar reciprocal of F with regard to V. Now since the equation in plane-coordinates of F is $\Sigma u_i^2=0$, then if

x is the pole with regard to U of a tangent plane u of F we shall have

$$\rho u_i = \frac{\partial U}{\partial x_i}, \quad i = 1, 2, 3, 4.$$

Hence, by the condition mentioned,

$$\Sigma \left(\frac{\partial U}{\partial x_i}\right)^2 \equiv K . \Sigma \left(\frac{\partial V}{\partial x_i}\right)^2,$$

where K is a constant depending on the coefficients of U and V.

Differentiating this identity we obtain

$$\sum_1^4 a_{i\kappa} \frac{\partial U}{\partial x_\kappa} \equiv K \sum_1^4 b_{i\kappa} \frac{\partial V}{\partial x_\kappa}, \quad i = 1, 2, 3, 4.$$

Now if $$\rho u_i = \frac{\partial U}{\partial x_i}, \quad \sigma v_i = \frac{\partial V}{\partial x_i},$$

then u and v are corresponding planes in the collineation determined by U and V, and hence from above

$$\sum_1^4 a_{i\kappa} u_\kappa = \rho' K \sum_1^4 b_{i\kappa} v_\kappa, \quad i = 1, 2, 3, 4.$$

This shows that u and v are *also corresponding planes in the collineation determined by the quadrics*

$$\Sigma a_{ik} u_i u_k = 0, \quad \Sigma b_{ik} u_i u_k = 0;$$

i.e. for the collineations determined by the quadrics U' and V' which are the *polar reciprocals* of U and V respectively with regard to F; and these latter quadrics are in point-coordinates

$$\Sigma A_{ik} x_i x_k = 0, \quad \Sigma B_{ik} x_i x_k = 0.$$

The invariants of the original pair of quadrics $\Sigma a_{ik} x_i x_k = 0$, $\Sigma b_{ik} x_i x_k = 0$ being denoted by Δ, Θ, Φ, Θ', Δ', it is easy to see that the invariants of this latter pair are

$$\Delta^3, \ \Theta' \Delta^2, \ \Delta \Delta' \Phi, \ \Theta \Delta'^2, \ \Delta'^3;$$

and the condition requires that the absolute invariants of the two collineations should be equal, *i.e.* that the following equations should hold

$$m \frac{\Theta}{\Delta} = n \frac{\Theta'}{\Delta}, \quad m^2 \frac{\Phi}{\Delta} = n^2 \frac{\Phi \Delta'}{\Delta^2}, \quad m^3 \frac{\Theta'}{\Delta} = n^3 \frac{\Theta \Delta'^2}{\Delta^3}, \quad m^4 \frac{\Delta'}{\Delta} = n^4 \frac{\Delta'^3}{\Delta^3}.$$

These are all satisfied by the single condition

$$\Delta \Theta'^2 = \Delta' \Theta^2, \text{ or } \iota_3^2 = \iota_1^2 \iota_4.$$

It is clear, denoting by f the polar reciprocal of F with regard to U, that f is also unaltered by the give ncollineation; both f and F belong to the infinite set of quadrics thus unaltered

ON FAMILIES OF QUADRATIC FORMS IN A GENERAL FIELD.

By Dr. L. E. Dickson.

1. IN an investigation, as yet unpublished, on the types of commutative linear groups in a general field (domain of rationality) F, I have shown that the problem of the normalization of the groups involves the normalization of families of quadratic or bilinear forms $\lambda_1 g_1 + \ldots + \lambda_m g_m$ by transformations in F both of the m parameters λ_i and the n variables entering the g_i. For the field C of all real and complex numbers, Jordan* has quite recently investigated the latter problem for certain sets of values of m and n. In addition to the simple case $n=2$, I treat in the present paper the case $m=2$, $n=3$, for a general field; viz., the case of two linearly independent ternary quadratic forms. The results are exhaustive for the field C, the field R of all real numbers, and any finite field. The main difficulty lies in the case in which the family contains no binary forms, and that in which the binary forms are all irreducible. Neither of these cases occur when the field is C or R, so that the problem is quite simple for these fields.

For C, there are 8 types†: ξ_1^2, $\xi_2^2 + 2\xi_1\xi_3$; and $\xi_1\xi_2$ with

$$\xi_1^2,\ \xi_1^2 - \xi_2^2,\ \xi_3^2,\ \xi_3^2 - \xi_1^2,\ \xi_3^2 - \xi_1^2 - \xi_2^2,\ \xi_1\xi_3,\ \xi_1\xi_3 + \xi_2^2.$$

For R, the 13 types are the preceding 8; ξ_1^2, $\xi_2^2 + \xi_3^2$; and

$$\xi_1\xi_2 \text{ with } \xi_3^2 + \xi_1^2 \pm \xi_2^2,\ \xi_1^2 + \xi_2^2,\ \xi_1^2 + \xi_3^2.$$

For the $GF[p^n]$, $p > 2$, there are 15 types‡,

$$\xi_1^2 - \xi_2\xi_3,\ \xi_1\xi_2 + \xi_2^2 + a\xi_3^2 \qquad (x^3 - 4ax + a \text{ irreducible}),$$

$$\xi_1^2 - v\xi_2^2,\ \xi_3^2 - r\xi_2^2 + 2s\xi_1\xi_2 \qquad (v \text{ and } r^2 - 4vs^2 \text{ not-squares}),$$

$$\xi_1^2 \text{ with } \xi_2^2 - v\xi_3^2 \text{ or } \xi_2^2 + 2\xi_1\xi_3,$$

$\xi_1\xi_2$ with one of the following, where $\rho = 0$, 1, or v:

$$\xi_3^2 - v\xi_1^2 - \xi_2^2,\ \xi_3^2 - v\xi_1^2 - v\xi_2^2,\ \xi_3^2 - \xi_1^2 - \xi_2^2,$$

$$\xi_3^2 - \rho\xi_1^2,\ \xi_1^2 - \rho\xi_2^2,\ \xi_1\xi_3,\ \xi_1\xi_3 + \xi_2^2.$$

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* *Journal de Mathématiques*, 1906, pp 403–438; 1907, pp. 5–51.

† These are equivalent to Jordan's types for $m=2$, $n=2$ and 3.

‡ Here a, v, r, s may be given fixed values in the field.

2. The maximum number of linearly independent binary forms is three. For $m=3$, the family contains ξ_1^2, $\xi_1\xi_2$, ξ_2^2.

Next let $m=n=2$, and let F be any field not having modulus 2. The family may be assumed to contain

$$\xi_1^2+k\xi_2^2,\ 2s\xi_1\xi_2+t\xi_2^2.$$

If $s\neq 0$, we may transform the second into $2\xi_1\xi_2$. Hence the family contains

$$(1)\qquad b_1=\xi_1^2+r\xi_2^2,\ b_2=2\xi_1\xi_2.$$

If $s=0$, we derive ξ_2^2, ξ_1^2. These may be transformed into $(\xi_1+\xi_2)^2$ and $(\xi_1-\xi_2)^2$, from which we derive a pair (1).

The determinant of xb_1+yb_2 is rx^2-y^2. The discriminant r of the latter is invariant up to a square factor when the variables ξ_1, ξ_2 and the parameters x, y are transformed linearly. *Hence two families* (1), *with the parameters* r_1, r_2, *are equivalent in* F *if, and only if,* $r_2=\rho^2 r_1$, *where* ρ *is an element* $\neq 0$ *of* F.

For $m=1$, $n=2$, the canonical type is $\xi_1^2+r\xi_2^2$, where, as before, r can be altered only by a square factor.

Families based on two linearly independent ternary quadratic forms in F are considered under cases I.–IV.

I. *Families of ternary quadratic forms containing no binary forms.*

3. First, let no form of the family be reducible to a binary form in the given field* F, not having modulus 2. The family may evidently be assumed to contain

$$\xi_3^2+f'_{12},\ a\xi_1\xi_3+b\xi_2\xi_3+f''_{12},$$

where the f_{12} are forms in ξ_1 and ξ_2. Since a and b are not both zero, we may apply a transformation on ξ_1 and ξ_2, and replace the second form by $\xi_2\xi_3+f'''_{12}$. Applying

$$\xi_3'=\xi_3+\rho\xi_1+\sigma\xi_2$$

to the latter, we reach $\xi_2\xi_3-c\xi_1^2$, $c\neq 0$. Multiply ξ_2 by c. The family now contains

$$\xi_1^2-\xi_2\xi_3,\ \xi_3^2+\delta\xi_1\xi_3+\epsilon\xi_2\xi_3+f_{12}.$$

The first is invariant under the transformation

$$\xi_1'=\xi_1+c\xi_2,\ \xi_2'=\xi_2,\ \xi_3'=\xi_3+2c\xi_1+c^2\xi_2.$$

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* Evidently neither C nor R.

For $c=-\frac{1}{4}\delta$, this replaces the second form by a similar one lacking $\xi_1\xi_3$. Adding a multiple of the first to eliminate $\xi_2\xi_3$, we get $\xi_3^2+g\xi_1^2+r\xi_2^2+s\xi_1\xi_2$. If $s=0$, the family contains an f_{23}. Hence we reach

$$(2)\quad g_1=\xi_1^2-\xi_2\xi_3,\ g_2=\xi_1\xi_2+a_1\xi_1^2+a_2\xi_2^2+a_3\xi_3^2\quad(a_3\neq 0).$$

The determinant of xg_1+yg_2 is $-\frac{1}{2}\Delta$, where

$$(3)\quad \Delta\equiv x^3+a_1x^2y-4a_2a_3xy^2+(a_3-4a_1a_2a_3)\,y^3;$$

Its first minors are relatively prime, since the family contains no binary form, Δ is irreducible in F. Its discriminant equals

$$(4)\quad 256a_2^3a_3^3-27a_3^2+144a_1a_2a_3^2-4a_1^3a_3 \\ +16a_1^4a_2a_3-128a_1^2a_2^2a_3^2.$$

4. Within the field F, let the transformation

$$(5)\qquad \xi_i=\sum_{j=1}^{3}d_{ij}\eta_j\quad(i=1,\,2,\,3),$$

replace g_1 and g_2 by, respectively,

$$(6)\qquad \lambda Q_1+\mu Q_2,\ \kappa Q_1+\sigma Q_2\quad(\lambda\sigma-\kappa\mu\neq 0),$$

where

$$(7)\quad Q_1=\eta_1^2-\eta_2\eta_3,\ Q_2=\eta_1\eta_2+A_1\eta_1^2+A_2\eta_2^2+A_3\eta_3^2\ (A_3\neq 0).$$

Then the following twelve relations hold:

$$(8)\quad D_{13}=0,\ D_{12}=\mu,\ D_{23}=-\lambda,\ D_1=\lambda+\mu A_1,\ D_2=\mu A_2,\ D_3=\mu A_3,$$

$$(9)\quad E_{13}=0,\ E_{12}=\sigma,\ E_{23}=-\kappa,\ E_1=\kappa+\sigma A_1,\ E_2=\sigma A_2,\ E_3=\sigma A_3,$$

in which

$$(10)\quad D_j=d_{1j}^2-d_{2j}d_{3j},\ D_{jk}=2d_{1j}d_{1k}-d_{2j}d_{3k}-d_{2k}d_{3j},$$

$$(11)\quad E_j=d_{1j}d_{2j}+\sum_{i=1}^{3}a_id_{ij}^2,\ E_{jk}=d_{1j}d_{2k}+d_{1k}d_{2j}+2\sum_{i=1}^{3}a_id_{ij}d_{ik}.$$

We proceed to express the d_{i1}, d_{i2} in terms of

$$(12)\qquad d_{13}=d,\ d_{23}=e,\ d_{33}=f.$$

Now d_{11}, d_{21}, d_{31} must be proportional to the determinants of their coefficients in $D_{13}=0$, $E_{13}=0$; viz.,

$$(13)\quad \begin{cases}\alpha_1=de+2a_2e^2-2a_3f^2,\ \alpha_2=-e^2-2a_1de-4a_3df,\\ \alpha_3=2d^2+ef+4a_2de+2a_1df.\end{cases}$$

The α_i are not all zero. Indeed,

$$-\alpha_1(4a_3^2f+2a_1a_3e)-\alpha_2a_3e$$

is of the form (3) for $x+2a_3f$, $y=e$. Hence we have

(14) $\quad d_{11}=\rho^{-1}\alpha_1,\ d_{21}=\rho^{-1}\alpha_2,\ d_{31}=\rho^{-1}\alpha_3 \quad (\rho\neq 0).$

Thus $D_{13}=0$ and $E_{13}=0$ give the useful identities

(15) $2\alpha_1d-d_2f-\alpha_3e\equiv 0;\ \alpha_1(e+2a_1d)+\alpha_2(d+2a_2e)+2a_3a_3f\equiv 0.$

By (6), σ and μ cannot both vanish. Hence

(16) $\quad D_3\equiv d^2-ef,\ E_3\equiv de+a_1d^2+a_2e^2+a_3f^2$

must not both vanish. Eliminating A_3 and A_1 from (8) and (9), we get

(17) $\quad \sigma D_3=\mu E_3,\ \sigma(D_1+D_{23})=\mu(E_1+E_{23}).$

But by $\mu=D_{12}$, $\sigma=E_{12}$, and (14),

(18) $\quad \rho\mu=2\alpha_1d_{12}-\alpha_3d_{22}-\alpha_2d_{32},$

(19) $\quad \rho\sigma=(\alpha_2+2a_1\alpha_1)d_{12}+(\alpha_1+2a_2\alpha_2)d_{22}+2a_3\alpha_3d_{32}.$

Hence (17_1) may be given the form

(20) $\quad R_1d_{12}+R_2d_{22}+R_3d_{32}=0,$

(21) $\quad \begin{cases} R_1=D_3(\alpha_2+2a_1\alpha_1)-2E_3\alpha_1, \\ R_2=D_3(\alpha_1+2a_2\alpha_2)+E_3\alpha_3,\ R_3=2D_3a_3\alpha_3+E_3\alpha_2. \end{cases}$

We may replace (17_2), which is quadratic in the d_{i2}, by

$$E_3(D_1+D_{23})-D_3(E_1+E_{23}),$$

and hence obtain

(22) $\quad S_1d_{12}+S_2d_{22}+S_3d_{32}+S=0,$

(23) $\quad \begin{cases} S=E_3D_1-D_3E_1,\ S_1=2E_3d-D_3(e+2a_1d), \\ S_2=-E_3f-D_3(d+2a_2e),\ S_3=-E_3e-2D_3a_3f. \end{cases}$

We thus have

(24) $\quad S=(\lambda\sigma-\kappa\mu)A_3,$

(25) $\quad \rho^2S=E_3(\alpha_1^2-\alpha_2\alpha_3)-D_3(\alpha_1\alpha_2+a_1\alpha_1^2+a_2\alpha_2^2+a_3\alpha_3^2)\equiv B.$

Since the determinants of the coefficients of E_3 and D_3 in the S_i are the determinants (13) of the coefficients in D_{12} and E_{13}, the S_i are not all zero. Likewise, the determinants in (21) are the corresponding functions of the α_i, so that the R_i are not all zero.

We shall make use of the identities

$$R_1 d + R_2 e + R_3 f \equiv 0, \tag{26}$$

$$S_1 d + S_2 e + S_3 f \equiv 0. \tag{27}$$

To prove them, we insert the values of the R_i, S_i. In (27) the coefficient of E_3 equals $2D_3$, that of D_3 equals $-2E_3$. In (26) the coefficients of $-E_3$ and D_3 are the expressions (15). We note the further identities

$$\alpha_3 e - \alpha_2 f \equiv 2S_1, \ \alpha_1 f - \alpha_3 d \equiv 2S_2, \ \alpha_2 d - \alpha_1 e \equiv 2S_3. \tag{28}$$

In (20) and (26), the R_i are proportional to the determinants (33) of their coefficients. Denote by m the non-vanishing ratio of these determinants to the R_i. Substitute the resulting values

$$f d_{22} = e d_{32} + m R_1, \ f d_{12} = d d_{32} - m R_2$$

in the product of (22) by f. The coefficient of d_{32} is zero by (27). Hence

$$m (R_2 S_1 - R_1 S_2) = fS.$$

Similarly, or directly by (26) and (27), we get

$$m (R_3 S_2 - R_2 S_3) = dS, \ m (R_1 S_3 - R_3 S_1) = eS.$$

In $R_3 S_2 - R_2 S_3$, we replace the R_i and S_i by their values (21) and (23), then apply (28), and finally replace the new S_i by their values. Hence

$$R_3 S_2 - R_2 S_3 = 4dK, \tag{29}$$

$$K \equiv E_3^3 - a_1 E_3^2 D_3 - 4 a_2 a_3 E_3 D_3^2 - (a_3 - 4 a_1 a_2 a_3) D_3^3. \tag{30}$$

Here $K \neq 0$ in view of the irreducibility of (3) and the assumption that E_3 and D_3 are not both zero. Similarly, or directly from (29), $S \neq 0$, $K \neq 0$, and the relations in m, we get

$$R_1 S_3 - R_3 S_1 = 4eK, \ R_2 S_1 - R_1 S_2 = 4fK. \tag{31}$$

In particular, we deduce

$$S = 4mK, \ B = 4m\rho^2 K,$$

employing (25). The latter must be an identity in d, e, f. But in B and K the coefficients of f^6 are $4a_3^3$ and a_3^3, respectively. Hence*

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* To indicate a direct proof of $B = 4K$, we modify B by replacing $a_2 a_3$ by $\frac{1}{2}a_2 a_3 + \frac{1}{2}a_3 a_2$, $a_1 a_2$ by $\frac{1}{2}a_1 a_2 + \frac{1}{2}a_2 a_1$, a_i^2 by $a_i a_i$, then replace the second factor in each $a_i a_j$ by its values from (13). The resulting expression may be written so that the a_i occur only in the combinations (28). Replacing the S_i by their values (23), we find that the expression reduces to $4K$.

$$(32) \qquad S=4\rho^{-2}K,\ B\equiv 4K,$$

$$(33) \quad \begin{vmatrix} d_{22} & e \\ d_{32} & f \end{vmatrix} = \rho^{-2}R_1, \quad \begin{vmatrix} d_{32} & f \\ d_{12} & d \end{vmatrix} = \rho^{-2}R_2, \quad \begin{vmatrix} d_{12} & d \\ d_{22} & e \end{vmatrix} = \rho^{-2}R_3.$$

In the next developments we assume that $d\neq 0$. The fact that all indicated divisions by d can be performed exactly would indicate that the resulting formulæ hold also for the case $d=0$; direct proof may be readily supplied. Then by (33_2), (33_3),

$$(33') \quad d_{32}=d^{-1}(fd_{12}+\rho^{-2}R_2),\ d_{22}=d^{-1}(ed_{12}-\rho^{-2}R_3).$$

Hence the fifth equations in (8) and (9) become

$$(34) \quad \rho^4 d^2\mu A_2=\rho^4 d_{12}{}^2 D_3+\rho^2 d_{12}\delta+R_2R_3,$$

$$(35) \quad \rho^4 d^2\sigma A_2=\rho^4 d_{12}{}^2 E_3+\rho^2 d_{12}\epsilon+a_3R_2{}^2+a_2R_3{}^2,$$

where

$$(36) \quad \delta\equiv R_3f-R_2e,\ \epsilon\equiv 2a_3fR_2-2a_2eR_3-dR_3.$$

Multiply (34) by $-E_3$, (35) by D_3, and add. The products on the left cancel in view of the final equations (8), (9). Further,

$$(37) \qquad D_3\epsilon-E_3\delta=R_3S_2-R_2S_3=4dK,$$

by (23), (29). Hence we have

$$(38) \quad 4dK\rho^2 d_{12}=E_3R_2R_3-D_3(a_3R_2{}^2+a_2R_3{}^2)\equiv U.$$

Inserting the values (21), we obtain for U the value

$$E_3{}^3\alpha_2\alpha_3-4a_2a_3D_3{}^2E_3\alpha_2\alpha_3-a_3D_3{}^3\alpha_1{}^2 + (E_3{}^2-4a_2a_3D_3{}^2)(\alpha_1\alpha_2+a_2\alpha_2{}^2+a_3\alpha_3{}^2)D_3.$$

We replace the final factor by its value from $B\equiv 4K$ in (25). The terms in $\alpha_2\alpha_3$ cancel, while the coefficient of $\alpha_1{}^2$ equals K, given by (30). Hence

$$(39) \qquad U=KV,\ V\equiv \alpha_1{}^3-4E_3{}^2+16a_2a_3D_3{}^2.$$

We find that V is a multiple of d. Hence

$$(40) \quad \rho^2 d_{12}=(4a_2a_3-a_1{}^2)d^3-2a_1d^2e-8a_2a_3def-2a_1a_3df^2 - (2a_1a_2+\tfrac{3}{4})de^2-3a_3ef^2-a_2e^3.$$

While we do not employ them later, we give for completeness the expressions for (33′), the divisions being exact:

$$(41) \quad \rho^3 d_{22}=-4a_3d^3+(a_1{}^2-4a_2a_3)d^2e+6a_3def+a_1de^2 + 4a_2a_3e^2f+4a_1a_3ef^2+\tfrac{1}{4}e^3+4a_3{}^2f^3,$$

$$(42)\quad \rho^2 d_{32} = 2a_1 d^3 + 3d^2 e + (a_1^2 - 4a_2 a_3)\, d^2 f + 6a_3 de^2 + a_1 def + 4a_2 a_3 ef^2 + (4a_1 a_2 - \tfrac{3}{4}) e^2 f + 4a_2^2 e^3.$$

In view of (33), (12), (14), the determinant of (5) is

$$\rho^{-3}(\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3).$$

Inserting the values (21) of the R_s and applying (25) and (32), we get

$$(43)\qquad |d_{ij}| = -8K\rho^{-3}.$$

Since we have determined the d_{i1} and d_{i2} in terms of the d_{i3}, so as to satisfy all the relations which result from (8) and (9) by the elimination of λ, μ, κ, σ and the A_{ij}, we may state the

THEOREM. *The various families of forms $xg_1 + yg_2$, defined by* (2) *and having irreducible determinants* (3) *in the field F, are transformed amongst themselves by precisely those transformations (d_{ij}) in F in which the d_{ij} are given by* (12), (14), (40)–(42), *d, e, f being any elements such that the functions* (16) *do not both vanish. Thus d, e, f shall not be a set of values of ξ_1, ξ_2, ξ_3 for which g_1 and g_2 both vanish.*

5. If $g_1 = g_2 = 0$, we may set

$$(44)\qquad \xi_1 = t\eta\zeta,\ \ \xi_2 = t\eta^2,\ \ \xi_3 = t\zeta^2,$$

where t, η, ζ are elements of F, $t \neq 0$, and

$$(45)\qquad \eta^3\zeta + a_1\eta^2\zeta^2 + a_2\eta^4 + a_3\zeta^4 = 0.$$

If $a_2 = 0$, then $\zeta = 0$, since the cubic factor is irreducible by (3). Then $\xi_1 = \xi_3 = 0$, while ξ_2 is arbitrary. Let next $a_2 \neq 0$. Except for the trivial case $\eta = \zeta = 0$, we may set $\zeta \neq 0$, $z = \eta/\zeta$. The resulting quartic in z is seen to have the resolvent cubic given by (3) for $y = -a_2^{-1}$. Hence the quartic has at most one root in the field F. If F is the field of the rational numbers, there is, in general, no root in F. But when F is a finite field, it follows,* from the irreducibility of (3), that the quartic has one and but one root r in the field. The solutions (44) are therefore $\xi_1 = r\xi_3$, $\xi_2 = r^2\xi_3$, ξ_3 arbitrary. Taking account of the parameter ρ, we have the

COROLLARY. *When F is the $GF[p^n]$, $p > 2$, the number of valid transformations is always $(p^{3n} - p^n)(p^n - 1)$.*

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* Dickson, *Bulletin, Amer. Math. Soc.*, October, 1906, p. 5, and middle of p. 7.

6. In (24) and elsewhere, we used the value of $\lambda\sigma - \kappa\mu$. Eliminating d_{22} and d_{32} from (18) and (19), by (33′), and applying (15), we get

$$\rho^3 d\mu = R_3\alpha_3 - R_2\alpha_2, \quad \rho^3 d\sigma = 2R_2\alpha_3\alpha_3 - R_3(\alpha_1 + 2a_2\alpha_2).$$

The second members are the products of D_3 and E_3 by

$$2a_3\alpha_3^2 - \alpha_1\alpha_2 - 2a_2\alpha_2^2 \equiv 4dT, \tag{46}$$

where

$$\begin{aligned} T = {} & 2a_3d^3 + 8a_2a_3d^2e + 4a_1a_3d^2f \\ & + (8a_2^2a_3 - 2a_1^2a_2 + \tfrac{1}{2}a_1)\,de^2 + 3a_3def + (2a_1^2a_3 - 8a_2a_3^2)\,df^2 \\ & + (\tfrac{1}{4} - a_1a_2)\,e^3 + 2a_2a_3e^2f + a_1a_3ef^2 - 2a_3^2f^3. \end{aligned} \tag{47}$$

Hence we have

$$\rho^3\mu = 4D_3T, \quad \rho^3\sigma = 4E_3T. \tag{48}$$

From the third equations in (8) and (9), we get

$$\rho^3 d\lambda = -\delta - 2\rho^2 d_{12}D_3, \quad -\rho^2 d\kappa = \varepsilon + 2\rho^2 d_{12}E_3, \tag{49}$$

where δ and ε are given in (36). Hence

$$\rho^5 d(\lambda\sigma - \kappa\mu) = 4T(D_3\varepsilon - E_3\delta).$$

Then, by (37),

$$\rho^5(\lambda\sigma - \kappa\mu) = 16TK. \tag{50}$$

We readily prove that T vanishes only when $d = e = f = 0$. First, for $d = 0$, $-4a_3T$ is of the form (3) for $x = 2a_3f$, $y = -e$, and hence vanishes only when $e = f = 0$. Let next $d \neq 0$. Then (46) is the determinant of the coefficients of f and e in relations (15). Hence if we multiply (15_1) by $2a_3\alpha_3$, (15_2) by α_2, and add, we find that $T = 0$ requires

$$2a_1\alpha_1\alpha_2 + \alpha_2^2 + 4a_3\alpha_1\alpha_3 = 0.$$

This expression and (46) are functions analogous to $-\alpha_2$ and $-\alpha_1$ in (13), so that they vanish only when $\alpha_2 = \alpha_3 = 0$. Then $\alpha_1 = 0$ by (15_1). But the d_i all vanish when $d = e = f = 0$.

Another proof follows from the fact that T, aside from a constant factor, equals the determinant of the triple algebra based on the field $F(i)$ defined by a root of (3). For instance, in the important case (see § 10) $a_1 = 0$, $a_2 = 1$, we find that (47) has the factor

$$2d + 2fi + a_3^{-1}ei^2 \quad (i^3 - 4a_3i + a_3 = 0). \tag{51}$$

7. We proceed to determine the A_i. We first modify the expressions (36). Applying (21), we get

$$\delta = E_3(\alpha_2 f - \alpha_3 e) + D_3(\alpha_2 d - \alpha_1 e) + D_3 g,$$

$$\epsilon = 2D_3 a_3(\alpha_1 f - \alpha_3 d) - 4D_3 a_2 a_3(\alpha_3 e - \alpha_2 f) + E_3 g,$$

where

$$g = 2a_3\alpha_3 f - \alpha_2 d - 2a_2\alpha_2 e. \tag{52}$$

Applying (28) and then (23), we get

$$\delta = D_3 G - 4d\beta, \quad \epsilon = E_3 G + 4d\gamma, \tag{53}$$

where

$$\begin{cases} G \equiv g - 4D_3 a_3 f, \ \beta \equiv E_3^2 - a_1 D_3 E_3, \\ \gamma \equiv D_3^2 a_3(4a_1 a_2 - 1) - 4D_3 E_3 a_2 a_3, \end{cases} \tag{54}$$

We now determine A_1, By (8) and (9),

$$(\lambda\sigma - \kappa\mu) A_1 = D_1 E_{23} - E_1 D_{23} = \lambda E_1 - \kappa D_1.$$

We multiply equations (49) by E_1 and D_1, respectively, and add. The coefficients of $2\rho^2 d_{12}$ equals S by (23). Thus, by (53),

$$\rho^2 d(\lambda E_1 - \kappa D_1) = 2\rho^2 d_{12} S + GS + 4dW,$$

$$W \equiv D_1\gamma + E_1\beta.$$

But (30) may be given the form

$$K = D_3\gamma + E_3\beta. \tag{55}$$

Hence, by (23),

$$D_3 W = D_1 K - \beta S.$$

Inserting the value of D_1 from (10) and (14), and applying (32), we get

$$4D_3 W = S(J + 4a_1 D_3 E_3),$$

where, by a direct factorization,

$$J \equiv \alpha_1^2 - \alpha_2\alpha_3 - 4E_3^2 = D_3\{8a_3 f(d + 2a_2 e) - (e + 2a_1 d)^2\}. \tag{56}$$

Hence if $D_3 \neq 0$, we obtain

$$4W = S(8a_3 df + 16a_2 a_3 ef + 4a_1 a_3 f^2 + (4a_1 a_2 - 1) e^2].$$

If $D_3 = 0$, we may compute $E_3 W$ similarly. Hence

$$\rho^2(\lambda\sigma - \kappa\mu) A_1 = SN, \tag{57}$$

$$N \equiv (8a_2 a_3 - 2a_1^2) d^2 - 2a_1 de + 12a_3 df + (4a_1 a_2 - \tfrac{3}{2}) e^2 + 16a_2 a_3 ef + 4a_1 a_3 f^2. \tag{58}$$

The determinant of N is $-2D$, D being given by (4).

To determine A_2, we multiply (34) by (49_2), (35) by (49_1), and add. We may omit the final terms of the multipliers (49), since the products by $-E_3$ and D_3 led to (38). Applying (37), (53), we get

$$(59)\quad \rho^6 d^3(\lambda\sigma-\kappa\mu)\,A_2 = 4dK\rho^4 d_{12}^{\ 2} + GU + 4dL,$$

where U is given by (38) or (39), while

$$(60)\qquad L \equiv R_2R_3\gamma + (a_3R_2^{\ 2} + a_2R_3^{\ 2})\,\beta.$$

We proceed to factor L. Applying (55) and (38), we get

$$D_3L = K(R_2R_3 - \beta V).$$

By (21) and (39),

$$R_2R_3 - \beta V = 2D_3^{\ 2}a_3\alpha_3(\alpha_1 + 2a_2\alpha_2) + a_1D_3E_3(16a_2a_3D_3^{\ 2} - 4E_3^{\ 2})$$
$$+ D_3E_3(\alpha_1\alpha_2 + a_1\alpha_1^{\ 2} + 2a_2\alpha_2^{\ 2} + 2a_3\alpha_3^{\ 2}) - 16\,a_2a_3D_3^{\ 2}E_3^{\ 2} - E_3^{\ 2}J.$$

The second member has the factor D_3 in view of (56). We may simplify the expression by replacing $D_3(\alpha_1\alpha_2+\ldots)$ by its values from (25), giving $B=4K$ its value from (30). After cancellations, we obtain D_3 times the quantity in brackets in (61). Hence if $D_3 \neq 0$,

$$(61)\quad L = K\{E_3(a_2\alpha_2^{\ 2} + a_3\alpha_3^{\ 2}) + 2a_3D_3\alpha_3(\alpha_1 + 2a_2\alpha_2) + 4a_3D_3^{\ 2}E_3\}.$$

We may obtain (61) directly from (60) by inserting the values of R_2, R_3, then subtracting the first four terms of (61). We get the product of $a_3D_3^{\ 2}E_3$ by the expression (25). The second member of (59) now has the factor $4dK$. The complementary factor is found to have the form d^2M, where

$$(62)\quad M = \{8a_1a_3 + (4a_2a_3 - a_1^{\ 2})^2\}\,d^4 + (4a_1^{\ 2}a_3 - 16a_2a_3^{\ 2})\,d^3f$$
$$+ (12a_3 - 8a_1a_2a_3 + 2a_1^{\ 3})d^3e + (30a_2a_3 - 16a_1a_2^{\ 2}a_3 + 4a_1^{\ 3}a_2 + \tfrac{1}{2}a_1^{\ 2})d^2e^2$$
$$+ (16a_1^{\ 2}a_2a_3 - 64a_2^{\ 2}a_3^{\ 2})\,d^2ef + (4a_1^{\ 3}a_3 - 16a_1a_2a_3^{\ 2})\,d^2f^2$$
$$+ (32a_2^{\ 2}a_3 + 4a_1^{\ 2}a_2 - \tfrac{1}{2}a_1)\,de^3 + (40a_1a_2a_3 - 9a_3)\,de^2f$$
$$+ 4a_1^{\ 2}a_3def^2 - 8a_1a_3^{\ 2}df^3$$
$$+ (16a_2^{\ 3}a_3 + a_1a_2 - \tfrac{3}{16})\,e^4 + (32a_1a_2^{\ 2}a_3 - 8a_2a_3)\,e^3f$$
$$+ (32a_2^{\ 2}a_3^{\ 2} + 16a_1^{\ 2}a_2a_3 - 3a_1a_3)e^2f^2 + (32a_1a_2a_3^{\ 2} - 12a_2^{\ 2})ef^3 + 16a_2a_3^{\ 3}f^4.$$

Hence we have

$$(63)\qquad \rho^6(\lambda\sigma-\kappa\mu)\,A_2 = 4KM.$$

Applying (24), (32), (50), (57), (63), *we obtain the results:*

$$(64)\quad A_1=\rho N/4T,\ A_2=\rho^{-1}M/4T,\ A_3=\rho^3/4T,$$

where N, T, M *are ternary forms in* d, e, f *of degrees* 2, 3, 4, *given by* (58), (47), (62).

8. For use below, we consider the case $a_1=a_2=0$. Then, by (3), $x^3+a_3y^3$ is irreducible in F, so that F does not have modulus 3. Then $D_3=E_3=0$ requires $d^3=d\cdot ef=-a_3f^3\cdot f$, so that $d=f=0$. Now

$$N=\tfrac{3}{2}(8a_3df-e^2),\quad M=3e(4a_3d^3-3a_3def-(e^3/16)-4a_3^2f^3).$$

Consider the transformations making $N=0$. To exclude the case $D_3=E_3=0$, we have merely to avoid the set $d=e=f=0$. The solutions in F of $N=0$ may be expressed by

$$d=rt^2,\quad f=rs^2/8a_3,\quad e=rst\quad (r\neq 0).$$

For these values, the second factor of M equals

$$4r^3(a_3t^3+\tfrac{1}{64}s^3)(t^3-s^3/8a_3),$$

and hence vanishes in F only when $t=s=0$. *Hence, if* $N=0$, *then* $M=0$ *if and only if* $e=0$.

In particular, $N=M=0$ requires $e=0$, $df=0$. Then

$$T=2a_3d^3-2a_3^2f^3.$$

If $d=0$, $f\neq 0$, then $A_3=a_3k^3$, $k=-\rho/2a_3f$. If $d\neq 0$, $f=0$, then $A_3=l^3/a_3$, $l=\rho/2d$. When F is a finite field, we may, in view of the first result, take $a_3=v$ or v^2, where v is a fixed not-cube. In view of the second result, we may replace $a_3=v^3$ by $A_3=v^3/a_3=v$. *In a finite field of order* $p^n=3m+1$, *the* $\frac{2}{3}(p^n-1)$ *families with* $a_1=a_2=0$ *are all equivalent. Each family has exactly* $3(p^n-1)$ *automorphs, given by* $d=e=0$, $f\neq 0$, $\rho=-2a_3f\omega$, *where* $\omega^3=1$.

9. Any family (2) with $a_2=0$ can be transformed into a family (7) with $A_2\neq 0$. If F does not have modulus 3, set $d=0$, $e=f=1$. Then

$$M=(-3/16a_3)(64a_3^2+16a_1a_3^2+a_2)\neq 0,$$

by (3) for $x=4a_3$, $y=1$. Also $D_3=-1$, so that the transformation is valid. If F has modulus 3, set $f=0$, $d=e=1$. Then

$$M=a_1(a_1^3-a_3-a_1^2-a_1+1)\neq 0,$$

by (3) for $x=1-a_1$, $y=-1$. In (7) with $A_2\neq 0$, we

multiply η_1 by A_2, η_3 by A_2^2, and obtain a similar family with $A_2=1$. *After a preliminary normalization of family* (2), *we have* $a_2=1$ *for* F *arbitrary.*

10. *Lemma. In the* $GF[p^n]$, $p>2$, *any family* (2) *is equivalent to a family with* $a_1=0$, $a_2=1$.

The ternary quadratic form N, given by (58), was shown to be of non-vanishing determinant. Hence (*Linear Groups*, p. 48), $N=0$ has exactly p^{2n} sets of solutions d, e, f in the $GF[p^n]$, $p>2$. By § 5, there are p^n sets d, e, f for which $D_3=E_3=0$. Hence these are valid transformations making $N=0$. Not all of them make $M=0$. For, if so, we may apply a preliminary normalization and set $a_1=a_2=0$. Then, by § 8, there are $p^{2n}-1$ sets d, e, f with D_3, E_3 not both zero, and making $N=0$, while only $2(p^n-1)$ of them make also $M=0$. Hence we may transform the family (2) into a family (7) with $A_1=0$, $A_2\neq 0$. As at the end of § 9, we may make $A_1=0$, $A_2=1$.

11. THEOREM. *In the* $GF[p^n]$, $p^n=3m+1$, *the various families* (2) *are all equivalent.*

In view of the remarks at the beginning of § 10, it suffices to prove the theorem for the families with $a_1=0$. Among them occur families with $a_1=a_2=0$, since there exist irreducible cubics $x^3+a_3y^3$ in the present field. To a definite family ϕ with $a_1=a_2=0$, we apply all transformations which result in families with $A_1=0$. By § 8, there are exactly $p^{2n}-1$ sets d, e, f for which $N=0$, while D_3 and E_3 are not both zero. Further, ρ may have any value $\neq 0$. Hence the total number of valid transformations is $(p^{2n}-1)(p^n-1)$. But ϕ has exactly $3(p^n-1)$ automorphs (§ 8). Hence ϕ is conjugate to exactly $\frac{1}{3}(p^{2n}-1)$ families with $A_1=0$. But for $a_1=0$, $y=1$, (3) is of the form

$$x^3+bx+a_3 \quad (b=-4a_2a_3). \tag{65}$$

The number of irreducible cubics of this type is exactly* $\frac{1}{3}(p^{2n}-1)$, since $p\neq 3$. Hence the theorem follows.

12. The preceding theorem holds also in the $GF[p^n]$, $p^n=3m+2$. We enlarge the field to $GF[p^{2n}]$ by adjoining a root of an irreducible quadratic equation, for example, $t^2=-3$. But a cubic (3) irreducible in the $GF[p^n]$ is irreducible also in the $GF[p^{2n}]$, since

$$a^{p^{2n}-1}=1,\ a^{p^{3n}-1}=1 \text{ imply } a^{p^n-1}=1.$$

* Dickson, *Bulletin*, l.c.

Also p^{2n} is of the form $3m+1$. Hence, by § 11, any two families (2) are equivalent in the $GF[p^{2n}]$, each being reducible in the $GF[p^{2n}]$ to a fixed family with $a_1=a_2=0$. Hence they may be shown to be equivalent in the $GF[p^n]$ by an argument* analogous to that used in proving that two transformations are equivalent in the $GF[p^n]$ if they have the same canonical form (in a higher field).

13. For the remaining case, the $GF[3^n]$, we determine directly the automorphs of a family (2) with $a_1=0$, $a_2=1$. Then $N=0$ requires $d^2=ef$, so that we set

$$e=rk^2,\ f=rm^2,\ d=rkm \quad (r\neq 0).$$

Set $a=a_3$. We find that

$$M=aE_3^2,\ E_3=r^2(am^4+mk^3+k^4). \tag{66}$$

By § 5, E_3 has a simple linear factor in the field,

$$L\equiv k-\sigma m, \tag{67}$$

where σ is the unique solution in the field of

$$\sigma^4+\sigma^3+a=0. \tag{68}$$

The complementary factor is

$$C=k^3+(1+\sigma)(k^2m+km^2\sigma+m^3\sigma^2). \tag{69}$$

Set $T=r^3\tau$. Then, by (47),

$$\tau=k^6-ak^5m+ak^4m^2-ak^3m^3+a^2km^5+a^2m^6.$$

In view of (68), we have the relation

$$\tau\equiv(\sigma^3+1)L^6-\sigma^3C^2 \pmod 3. \tag{70}$$

By (64), the condition for an automorph is

$$M^3\equiv aT^4 \pmod 3,$$

viz.

$$a^3L^6C^6\equiv a\tau^4.$$

Since $a^3\equiv-(\sigma^3+1)\sigma^9$, this may be written in the form

$$a\tau^4+x^2(\tau+x)=0,\ x\equiv\sigma^3C^2.$$

Since this function is of the same type as E_3, we have

$$x=\sigma\tau,\ \text{whence}\ \tau=\sigma^3C^2.$$

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* *Linear Groups*, p. 229.

Replacing τ by the above value, we get

$$\sigma C \equiv \pm(\sigma+1)L^3 \pmod 3.$$

The upper sign is excluded since the resulting cubic is irreducible, being derived from $x^3-axy^2+ay^3$, viz. (3), by the transformation, of determinant $\sigma^{-\frac{1}{3}}$,

$$x=-\sigma^{-\frac{2}{3}}k+\sigma^{\frac{1}{3}}(\sigma+1)m,\ y=-\sigma^{-\frac{5}{3}}k+\sigma^{-\frac{2}{3}}m.$$

For the lower sign, the equation may be written

$$ak^3=(\sigma+\sigma^2)^2L^3k. \tag{71}$$

Now a is square, since (3) is irreducible (*Bulletin*, l.c.). Hence, in addition to the solution $k=0$, m arbitrary $\neq 0$, there are two values in the field for the ratio m/k, k having any value $\neq 0$. *For the $GF[3^n]$ the number of automorphs of a family* (2), *with* $a_1=0$, $a_2=1$, *is exactly* $3(3^n-1)$.

The number of sets of solutions d, e, f of $N=0$, for which $E_3\neq 0$ (here $D_3=0$ when $N=0$), has been shown to be $3^{2n}-3^n$. Since ρ has 3^n-1 values, the number of valid transformations is $(3^{2n}-3^n)(3^n-1)$. Hence each family with $a_1=0$ is one of $\frac{1}{3}(3^{2n}-3^n)$ conjugates. But (*Bulletin*, l.c.) the total number of irreducible cubics (65) in the $GF[3^n]$ is $\frac{1}{3}(3^{2n}-3^n)$. *Hence the families are all conjugate.*

II. *Families of ternary quadratic forms containing binary forms all of which are irreducible in the field F.*

14. We may assume that the family contains

$$\xi_1^2-v\xi_2^2,\ a\xi_1^2+b\xi_3^2+c\xi_1\xi_2+d\xi_1\xi_3+e\xi_2\xi_3,$$

where v is a not-square in F. If $b=0$, we apply

$$\xi_3=\xi_3'+\rho\xi_1'+\sigma\xi_2',\ \xi_1=\xi_1',\ \xi_2=\xi_2', \tag{72}$$

to the second, then add $v^{-1}e\sigma(\xi_1^2-v\xi_2^2)$, and obtain

$$(a+d\rho+v^{-1}e\sigma)\xi_1^2+(c+e\rho+d\sigma)\xi_1\xi_2+d\xi_1\xi_3+e\xi_2\xi_3.$$

The coefficients of ξ_1^2 and $\xi_1\xi_2$ can be made to vanish by choice of ρ, σ, unless the determinant $d^2-v^{-1}e^2$ vanishes. In the latter case, $d=e=0$, and ξ_1 is a factor. In the contrary case, ξ_3 is a factor. In either case our hypothesis is contradicted. Hence $b\neq 0$. Again applying (72), we may make $d=e=0$. Hence the family contains

$$g_1=\xi_1^2-v\xi_2^2,\ g_2=\xi_3^2-r\xi_2^2+2s\xi_1\xi_2. \tag{73}$$

The determinant of xg_1+yg_2 equals $-yQ$, where

$$Q=vx^2+rxy+s^2y^2. \tag{74}$$

Thus the family contains a single binary form if

$$D\equiv r^2-4vs^2 \tag{75}$$

is a not-square in F; three binary forms if D is a square. The case $D=0$ is excluded, since then $r=s=0$ so that g_2 is a reducible binary form.

15. First, let $D=d^2$. Then $Q=0$ for

$$x=y(\pm d-r)/2v.$$

The binary forms other than g_1 are therefore

$$\xi_3^2+2s\xi_1\xi_2+\{(\pm d-r)/2v\}\,\xi_1^2-\{\tfrac{1}{2}(\pm d+r)\}\,\xi_2^2,$$

the last three terms forming a perfect square. Let first $s\neq 0$, so that $d^2\neq r^2$. A multiple of the form is

$$[s\xi_1-\{\tfrac{1}{2}(\pm d+r)\}\,\xi_2]^2-\{\tfrac{1}{2}(\pm d+r)\}\,\xi_3^2.$$

For each sign, this must be irreducible in F. Hence

$$\tfrac{1}{2}(r-d),\ \tfrac{1}{2}(r+d)$$

must be not-squares in F. Their product is vs^2, a not-square. But for the field of all real numbers, and for every finite field, the product of two not-squares is a square. For these fields the present case is excluded. Next, if $s=0$, the binary forms are g_1, g_2, and $\xi_3^2-v^{-1}r\xi_1^2$, so that v, r, $v^{-1}r$ must all be not-squares. Since the case $D=$square is excluded for finite fields (and for the field of all reals), we do not investigate it further.

16. Let next D be a not-square* in F. Since g_1 is the only binary form in the family, it must be transformed into a multiple of itself if the family (73) is transformed into a family of the same type. The transformation is therefore of the form

$$\xi_1=\alpha\eta_1+\beta\eta_2,\ \xi_2=\pm v^{-1}\beta\eta_1\pm\alpha\eta_2,\ \xi_3=\gamma\eta_3. \tag{76}$$

To the new form of g_2 we add a multiple of g_1 to eliminate η_1^2, and get

$$Q_2\equiv\gamma^2\eta_3^2-R\eta_2^2+2S\eta_1\eta_2,$$

* This case does not arise when F is the field of all reals.

where

$$(77)\quad R=r\{\alpha^2+(\beta^2/v)\}\mp 4\alpha\beta s,\quad S=\pm s\{\alpha^2+(\beta^2/v)\}-(\alpha\beta r/v).$$

Since the coefficients of Q_2 are homogeneous in α, β, γ, we may take $\gamma=1$ without loss of generality. The question is now the extent of the specialization possible for R and S, by choice of α and β in F.

The answer is immediate when F is the $GF[p^n]$, $p^n=4m+1$, so that -1 is a square. The discriminant of R is then a square, $-v^{-1}D$, so that by a preliminary (above) transformation we may set $r=0$. Applying now the above transformation with the upper signs, we have

$$R=-4\alpha\beta s,\quad S=s(\alpha^2+v^{-1}\beta^2).$$

If s is a square, we take $\beta=0$, $\alpha^2=s^{-1}$. If s is a not-square, we take $\alpha=0$, $\beta^2=vs^{-1}$. In each case, $R=0$, $S=1$.

For the $GF[p^n]$, $p^n=4m+1$, each family is equivalent to

$$(78)\qquad \xi_1^2-v\xi_2^2,\quad \xi_3^2+2\xi_1\xi_2,$$

where v is a particular not-square.

17. We proceed to determine the automorphs of a family (73) in which* D is a not-square in the $GF[p^n]$. We seek the solutions α, β of $R=r$, $S=s$. Now

$$Rs\mp Sr=\pm v^{-1}\alpha\beta D.$$

For the upper signs, we have $\alpha\beta=0$. If $\beta=0$, then $\alpha^2=1$. If $\beta\neq 0$, then $r=r\beta^2v^{-1}$, $s=s\beta^2v^{-1}$, by (77), whence $r=s=0$, contrary to $D\neq 0$.

For the lower signs, we have

$$(79)\qquad 2rs=-v^{-1}\alpha\beta D.$$

By (75), $s\neq 0$. If $r=0$, the conditions are

$$\alpha\beta=0,\quad -1=\alpha^2+\beta^2v^{-1},$$

and have just two sets of solutions in a finite field.

Next, let $r\neq 0$. Eliminating β between (77_1) and (79), and applying (75), we get

$$[2\alpha^2D-(r^2+4vs^2)]^2=D^2,\quad \alpha^2D=r^2 \text{ or } 4vs^2.$$

Since D is a not-square, we obtain two sets of solutions α, β in a finite field. *Hence there are in all four automorphs with $\gamma=1$.* The number of transformations (76) with $\gamma=1$ in the

* The analysis applies to other cases as well.

$GF[p^n]$, $p>2$, is $2(p^{2n}-1)$. Hence each family is one of $\frac{1}{2}(p^{2n}-1)$ conjugates. For a given value $\neq 0$ of D, there are p^n+1 sets of solutions of (75), by *Linear Groups*, p. 46. Hence there are exactly $\frac{1}{2}(p^n-1)(p^n+1)$ sets r, s for which D is a not-square.

THEOREM. *In any $GF[p^n]$, $p>2$, the various families (73), in which all the binary forms are irreducible, are equivalent. This case does not occur for the field of all real numbers.*

III. *Families whose reducible binary forms are perfect squares.*

18. The family is assumed to contain

$$\xi_1^2,\ a\xi_2^2+b\xi_3^2+c\xi_1\xi_2+d\xi_1\xi_3+e\xi_2\xi_3,$$

but no form L_1L_2, $L_1\neq L_2$. The case $a=b=0$ is excluded by the second form if $e=0$, and for $e\neq 0$ by

$$cd\xi_1^2+e(c\xi_1\xi_2+d\xi_1\xi_3+e\xi_2\xi_3)=(c\xi_1+e\xi_3)(d\xi_1+e\xi_2).$$

Hence we may set $a=1$. Completing the square in the ξ_2 terms, we obtain

(80) $$\xi_1^2,\ \xi_2^2-r\xi_3^2+2s\xi_1\xi_3.$$

If $r\neq 0$, we apply $\xi_3'=\xi_3+r^{-1}s\xi_1$ and get

(81) $$\xi_1^2,\ \xi_2^2-r\xi_3^2 \quad (r=\text{not-square}).$$

If $r=s=0$, we reach $\xi_1^2-\xi_2^2$, contrary to hypothesis. Finally, if $r=0$, $s\neq 0$, we multiply ξ_3 by s^{-1} and get

(82) $$\xi_1^2,\ \xi_2^2+2\xi_1\xi_3.$$

The families (81) and (82) are not equivalent in view of their determinants; neither contains a form $L_1L_2\neq L_1^2$. We can alter r only by a square factor.

IV. *Families containing a form reducible to $\xi_1\xi_2$.*

19. In the second form $a\xi_3^2+\ldots$, let first $a\neq 0$. Applying a transformation (72), we reach

(83) $$g_1=\xi_1\xi_2,\ g_2=\xi_3^2-r\xi_1^2-s\xi_2^2.$$

The determinant of $2xg_1+yg_2$ equals $y(rsy^2-x^2)$.

(i) First, let rs be a not-square in F. Then $\xi_1\xi_2$ is the only form reducible to a binary one. It therefore suffices to consider transformations which multiply $\xi_1\xi_2$ by a constant.

Hence (83) is equivalent only to families, of like type, in which r and s have been interchanged or multiplied by square factors.

(ii) Next, let rs be a square $\neq 0$. Then $s = rt^2$. Multiplying ξ_2 by t^{-1}, we may set $s = r$ in (83). The reducible forms are g_1 and

$$g_2 \pm 2rg_1 \equiv \xi_3^2 - r(\xi_1 \mp \xi_2)^2.$$

Hence if r is a not-square, a transformation normalizing (83) with $s = r$ must multiply $\xi_1\xi_2$ by a constant, and thus can only alter r by a square factor. But if r is a square, we may set $r = 1$.

(iii) Finally, let $rs = 0$. As g_2 we may take

$$\xi_3^2, \ \xi_3^2 - \xi_1^2, \text{ or } \xi_3^2 - r\xi_1^2 \quad (r = \text{not-square}).$$

where r can be altered only by a square factor.

20. Next, let $a = 0$. If ξ_3 is entirely absent, we have

$$\xi_1\xi_2, \ \xi_1^2 - r\xi_2^2. \tag{84}$$

In the contrary case, we may take

$$\xi_1\xi_2, \ \xi_1\xi_3 + b\xi_2\xi_3 + c\xi_1^2 + d\xi_2^2.$$

Applying $\xi_3 = \xi_3' - c\xi_1' + s\xi_2'$, we reach

$$\xi_1\xi_2, \ \xi_1\xi_3 + b\xi_2\xi_3 + (d + sb)\,\xi_2^2.$$

If $b = 0$, we multiply ξ_3 and obtain

$$\xi_1\xi_2, \ \xi_1\xi_3 + \xi_2^2 \text{ or } \xi_1\xi_3. \tag{85}$$

If $b \neq 0$, we make $d + sb = 0$, multiply ξ_2, and get

$$\xi_1\xi_2, \ \xi_1\xi_3 + \xi_2\xi_3.$$

Applying $\xi_1 = \eta_1 - \eta_3$, $\xi_2 = \eta_1 + \eta_3$, $\xi_3 = \eta_2$, we get

$$\eta_1^2 - \eta_3^2, \ 2\eta_1\eta_2,$$

a special case of (83).

The two families (85) are not equivalent, and neither is equivalent to a family (84), nor to one obtained in § 19, as shown by a consideration of the determinant and the greatest common divisor of its first minors.

For a detailed summary for various fields, see § 1.

The University of Chicago,
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SINGULAR LINES AND CURVES ON SURFACES.

By A. B. Basset, M.A., F.R.S.

1. THE object of the following paper is to discuss the theory of singular lines and singular plane curves on surfaces. Nodal, cuspidal, and triple lines and also nodal and cuspidal conics have been considered by various writers in connection with quartic surfaces; but a more general theory is necessary. For just as most compound singularities, which a cubic surface can possess, appear on such a surface in a special form; so various singularities, which a quartic surface can possess, are deficient in certain peculiarities which appear when the singularity occurs on a surface of higher degree.

The tangent planes at any point on a multiple line are in general *torsal** tangent planes, since they rotate or twist round the line as the point moves along it. It is, however, possible for one or more tangent planes to be fixed in space, and such lines usually possess certain peculiarities which necessitate a different mathematical treatment from that required when all the tangent planes are fixed. This is always the case with singular lines of the second order.

When a determinate number of nodes and cusps move up to coincidence along a plane curve, the compound singularity thereby formed is called a *singular point of the second order*. Such singularities are of the tacnode type, and any arbitrary line through the singular point has bitactic contact thereat; also there is only one distinct tangent at the point. When a straight line lying in a surface is such that any plane section possesses a singular point of this character, where the line cuts the plane, the former is called a *singular line of the second order*. There are two distinct species of such lines. In the first species the tangent plane is *torsal*; in the second it is *fixed* in space.

Surfaces may also possess singular lines and curves of any order†; and the curves may either be plane or twisted. In the present paper I shall confine my attention to certain

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* Cayley calls a line which possesses a *fixed* tangent plane a *torsal* line. This is a singularly inappropriate definition, since the word *torsal* is derived from *torsi* the perfect of *torquere* to twist; whereas the essential feature of such a line is that the tangent plane is devoid of twisting.

† To prevent mis-conception, I wish to state that I always call the surface $(\alpha, \beta, \gamma, \delta)^n = 0$ a surface of the nth *degree* and not a surface of the nth *order*; and similarly for curves.

singular lines and plane curves of the second order; and shall also discusss the case of a triple line, that is to say, a line such that any plane section of the surface has a triple point where the line intersects it.

References to my former papers on "Compound Singularities of Curves," vol. XXXVII., p. 313; "Singularities of Surfaces," vol. XXXVIII., p. 63 and 159; and "Multiple Points on Surfaces," vol. XXXIX., p. 1, will be respectively denoted by the letters S.C., S.S., and M.P.

Cuspidal Lines.

2. The general equation of a surface having a cuspidal line of the first species is

$$(L\alpha\gamma + M\beta\delta)^2(\alpha, \beta)^{n-4} + (P, Q, R, S\chi\gamma, \delta)^3 = 0 \ldots (1),$$

where P, Q, R, S are quarternary quantics of $(\alpha, \beta, \gamma, \delta)$ of degree $n-3$. This equation when written out at full length is

$$(L\alpha\gamma + M\beta\delta)^2(p_0\alpha^{n-4} + p_1\alpha^{n-5}\beta + \ldots p_{n-4}\beta^{n-4}) + \alpha^{n-3}v_3 + \alpha^{n-4}(\beta V_3 + V_4) + \ldots + \beta^{n-3}w_3 + \beta^{n-4}w_4 + \ldots w_n = 0 \ldots\ldots (2).$$

Cuspidal lines possess two kinds of singular points which occur (i) when the cusp changes into a tacnode, (ii) when there are cubic nodes on the line. I have shown* in S.S., §§ 43 and 44 that there are n points of the first species; but as (1) and (2) give the surface in a simplified form, I shall repeat the proof. Let the plane $\alpha = \lambda\beta$ cut AB in B', then the equation of the section is

$$\beta^{n-2}(L\lambda\gamma + M\delta)^2(p_0\lambda^{n-4} + p_1\lambda^{n-5} + \ldots p_{n-4}) + \beta^{n-3}(\lambda^{n-3}v_3 + \lambda^{n-4}V_3 + \ldots w_3) + \ldots = 0 \ldots\ldots (3),$$

which, for brevity, we shall write in the form

$$A\beta^{n-2}\Omega_1^2 + B\beta^{n-3} + \ldots = 0 \ldots\ldots\ldots\ldots (4).$$

3. *The cuspidal line possesses n tacnodal points, and n − 4 cubic nodes, at which there is a cuspidal cubic cone.*

The condition for a tacnodal point is that Ω_1 should be a factor of B, which requires that the eliminant of Ω_1 and B should vanish. This furnishes an equation of the n^{th} degree in λ.

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* In this investigation the existence of the cubic nodes was overlooked.

The points where the cuspidal line cuts the planes $(\alpha, \beta)^{n-4}=0$ are cubic nodes on the line, and there are $n-4$ of them. If A be one of these points $p_0=0$, and the coefficient of α^{n-3} equated to zero gives

$$L^2 p_1 \beta\gamma^2 + v_3 = 0,$$

which shows that AB is a cuspidal generator of the cone. A cuspidal line on a quartic surface has 4 tacnodal points, but no cubic nodes; hence on such a surface the line appears in an incomplete form.

4. *A cuspidal line of the second species possesses $n-3$ tacnodal points and $n-2$ cubic nodes.*

If γ be the fixed tangent plane, the equation of the surface must be of the form

$$\gamma^2 (\alpha, \beta)^{n-2} + (P, Q, R, S\chi\gamma, \delta)^3 = 0 \ldots\ldots\ldots\ldots (5).$$

Proceeding as in § 2, the first term of the equation corresponding to (3) must be of the form

$$\beta^{n-2}\gamma^2 \{p_0\lambda^{n-2} + p_1\lambda^{n-3} + \ldots p_{n-2}\},$$

and the condition for a cubic node is that this should vanish, which furnishes an equation of degree $n-2$ in λ. The condition for a tacnodal point is that the coefficient of δ^3 in the expression

$$\lambda^{n-3} v_3 + \lambda^{n-4} V_3 + \ldots w_3$$

should vanish, which furnishes an equation of degree $n-3$ in λ. A quartic surface having a cuspidal line of this character possesses both species of singular points.

Tacnodal Lines.

5. The general equation of a surface having a tacnodal line of the first species is

$$(L\alpha\gamma + M\beta\delta)^2 (\alpha, \beta)^{n-4} + 2 (L\alpha\gamma + M\beta\delta) (F, G, \ldots\chi\alpha, \beta)^{n-4} + (P, Q, R, S, T\chi\gamma, \delta)^4 = 0 \ldots\ldots\ldots (6),$$

where F, G, ... are binary quadrics of (γ, δ), and P, Q, R, S, T are quaternary quantics of $(\alpha, \beta, \gamma, \delta)$. Equation (6), when written out at full length, becomes

$$\begin{aligned}(L\alpha\gamma + M\beta\delta)^2 &(p_0\alpha^{n-4} + p_1\alpha^{n-5}\beta + \ldots p_{n-4}\beta^{n-4}) \\ &+ 2 (L\alpha\gamma + M\beta\delta) (\alpha^{n-4} v_2 + \alpha^{n-5}\beta V_2 + \ldots) \\ &+ \alpha^{n-4} w_4 + \alpha^{n-5} (\beta W_4 + W_5) + \ldots W_n = 0 \ldots\ldots\ldots\ldots (7).\end{aligned}$$

Tacnodal lines possess four species of singular points, viz. (i) points where the tacnode changes into a rhamphoid cusp; (ii) points where the tacnodal tangent has quinquetatic contact with the surface; (iii) points which are the special kind of tacnodes formed by making the two tangents at a biflecnode coincide; (iv) points which are cubic nodes on the surface. The first and fourth species of singular points are the only ones which affect the class of the surface.

6. Putting $\alpha = \lambda\beta$ in (7), the equation of the section by the plane $B'CD$ is

$$\beta^{n-2}(L\lambda\gamma + M\delta)^2(p_0\lambda^{n-4} + p_1\lambda^{n-3} + \ldots p_{n-4})$$
$$+ 2\beta^{n-3}(L\lambda\gamma + M\delta)(\lambda^{n-4}v_2 + \lambda^{n-5}v_1 + \ldots)$$
$$+ \beta^{n-4}(\lambda^{n-4}w_4 + \lambda^{n-5}W_4 + \ldots) + \ldots = 0 \quad \ldots\ldots\ldots\ldots (8),$$

which, for brevity, we shall write in the form

$$A\beta^{n-2}\Omega_1^2 + 2B\beta^{n-3}\Omega_1 + C\beta^{n-4} + \ldots = 0 \ldots\ldots\ldots (9).$$

(i) *There are* $2n-4$ *points where the tacnode changes into a rhamphoid cusp.*

The equation of a curve having a rhamphoid cusp at B' is

$$\beta^{n-4}(\beta\Omega_1 + \Omega_2)^2 + \beta^{n-4}\Omega_1 w_3 + \beta^{n-5}w_5 + \ldots w_n = 0 \ldots (10),$$

where $\Omega_n \equiv w_n \equiv (\gamma, \delta)^n$. The condition that (9) should be of this form is that Ω_1 should be a factor of $AC - B^2$; and since the eliminant is of degree $2n-4$ in λ, there are $2n-4$ of such points.

(ii) *There are* n *points where the tacnodal tangent has quinquetactic contact with the curve.*

The condition is that Ω_1 should be a factor of C, and the eliminant is of degree n in λ. This singularity has been discussed in S.C., § 4, (ii), and its point constituents are the same as those of an ordinary tacnode, but its line constituents are different. Its existence does not therefore affect the class of the surface. Let B' be one of these singular points, and let $B'E$ be any line drawn in the tangent plane at B'; then $B'E$ has quinquetactic contact with the surface at B'; but since $B'E$ has one degree of freedom, one or more positions must exist in which $B'E$ has sextactic contact with the surface. Accordingly, when the surface is a quintic, it must possess *at least* five lines lying in the suaface, which pass through the tacnodal line.

(iii) *There are $n-2$ points where the tacnode changes into the special one above mentioned.*

The condition is that Ω_1 should be a factor of B, and the eliminant is of degree $n-2$ in λ; also since the point constituents of the singularity are the same as those of an ordinary tacnode, it does not affect the class of the surface.

(iv) *There are $n-4$ points which are cubic nodes on the section, such that the nodal cone consists of a quadric cone and a plane touching the latter along the line AB.*

Let A be one of these points, then $p_0=0$ in (7); and the coefficient of α^{n-3} equated to zero furnishes the equation

$$(Lp_1\beta\gamma+2v_2)\gamma=0,$$

which is the cone in question. The existence of these points affects the class of the surface; also they cannot occur on a quartic surface.

7. *A tacnodal line of the second kind possesses $2n-6$ rhamphoid cuspidal points, and $n-2$ cubic nodes.*

The equation of the surface is

$$\gamma^2(\alpha,\beta)^{n-2}+2\gamma(\alpha^{n-3}v_2+\alpha^{n-4}\beta V_2+\ldots)$$
$$+\alpha^{n-4}w_4+\alpha^{n-5}(\beta W_4+W_5)+\ldots=0\ldots\ldots(11),$$

and the theorem can be proved as before.

Rhamphoid Cuspidal Lines.

8. The equation of a surface having a rhamphoid cuspidal line of the first kind is

$$(L\alpha\gamma+M\beta\delta+\Omega_1)^2(\alpha,\beta)^{n-4}+(L\alpha\gamma+M\beta\delta)(\alpha^{n-5}v_3+\alpha^{n-6}\beta V_3+\ldots)$$
$$+(P,Q,R,\ldots)(\gamma,\delta)^5=0\ldots.(12),$$

where $\Omega_2=(\gamma,\delta)^2$, and P, Q, ... are quarternary quantics of $(\alpha,\beta,\gamma,\delta)$.

Rhamphoid cuspidal lines possess two kinds of singular points which affect the class of the surface, and occur (i) when the rhamphoid cusp changes into an oscnode, (ii) when there are cubic nodes on the surface. Points also exist where the cuspidal tangent has quinquetatic contact with the surface, but points of this character will not be considered since they do not affect the class of the surface.

9. We must first find the equation of a plane curve of the n^{th} degree which has an oscnode at A.

When the singularity is a rhamphoid cusp, the equation of the curve is

$$\alpha^{n-4}(\alpha\gamma - u_2)^2 + \alpha^{n-4}\gamma u_3 + \alpha^{n-5}u_5 + \ldots = 0 \ldots\ldots (13),$$

where $u_n = (\beta, \gamma)^n$. If A is an oscnode, the conic $\alpha\gamma = u_2$ must have sextactic contact with (13) at A; hence eliminating α, (13) becomes

$$u_2^{n-4}u_3 + u_2^{n-5}u_5 + \gamma u_2^{n-6}u_6 + \ldots \gamma^{n-5}u_n = 0,$$

which determines the $2n-5$ ordinary points in which the conic cuts (13). The required condition is therefore that γ should be a factor of

$$u_2u_3 + u_5 \ldots\ldots\ldots\ldots\ldots\ldots (14).$$

It can also be shown that the conic has tritactic contact with the first polar of (13), hence all the necessary conditions are satisfied. Let l, p, P be the coefficients of the highest powers of β in u_2, u_3, u_5; and let $S = \alpha\gamma - u_2$; then (13) may be put into the form

$$\alpha^{n-4}(S^2 + \gamma^2 v_2) + \alpha^{n-5}(pS\beta^3 + \gamma v_4) + \alpha^{n-6}u_6 + \ldots u_n = 0 \ldots (15).$$

10. *There are $n-4$ cubic nodes, and n points where the rhamphoid cusp changes into an oscnode.*

The equation of the section of (12) by the plane $\alpha = \lambda\beta$ is

$$\beta^{n-4}(\beta\Omega_1 + \Omega_2)^2(\lambda, 1)^{n-4} + \beta^{n-4}\Omega_1(\lambda^{n-5}v_3 + \lambda^{n-6}v_3 + \ldots) + \beta^{n-5}(\lambda^{n-5}v_5 + \lambda^{n-6}W_5 + \ldots) + \ldots = 0,$$

where $\Omega_1 = L\lambda\gamma + M\delta$. Writing this in the form

$$\beta^{n-4}(\beta\Omega_1 + \Omega_2)^2(\lambda, 1)^{n-4} + A_3\beta^{n-4}\Omega_1 + A_5\beta^{n-5} + \ldots = 0,$$

it follows that the condition for an oscnodal point is that Ω_1 should be a factor of $A_5 - \Omega_2 A_3$, which furnishes an equation of the n^{th} degree for determining λ. The cubic nodes are determined by the equation $(\lambda, 1)^{n-4} = 0$; and the equation of the cone is

$$p_1 L\beta\gamma^5 = 0,$$

which shows that the cubic node is of the fifth species.

11. We must now find the equation of a surface having a rhamphoid cuspidal line of the second kind.

Consider the surface

$$(\alpha\gamma + \Omega_2)^2(\alpha, \beta)^{n-4} + \gamma^2(\alpha^{n-3}v_1 + \alpha^{n-4}\beta w_1 + \ldots) + \gamma(\alpha^{n-4}v_3 + \alpha^{n-5}\beta w_3 + \ldots) + \alpha^{n-5}w_5 + \alpha^{n-6}(\beta W_5 + W_6) + \ldots W_n = 0 \ldots\ldots\ldots (15),$$

where all the suffixed letters denote binary quantics of (γ, δ). Putting $\alpha = \lambda\beta$, (15) becomes

$$\beta^{n-4}(\lambda\beta\gamma + \Omega_2)^2(\lambda, 1)^{n-4} + \beta^{n-3}\gamma^2(\lambda^{n-3}v_1 + \lambda^{n-4}w_1 + \ldots) + \beta^{n-4}\gamma(\lambda^{n-4}v_2 + \lambda^{n-5}w_2 + \ldots) + \ldots = 0 \ldots\ldots\ldots\ldots (16),$$

and I have shown* that this is one of the forms of a curve which has a rhamphoid cusp at B'.

12. *There are $n-4$ cubic nodes and $n-3$ osculodal points on the line.*

The cubic nodes are given by the roots of the equation $(\lambda, 1)^{n-4} = 0$, whilst the condition for an osculodal point is that γ should be a factor of

$$\Omega_2^2(\lambda^{n-3}v_1 + \lambda^{n-4}w_1 + \ldots) + \Omega_2(\lambda^{n-4}v_2 + \ldots) + \lambda^{n-5}w_2 + \ldots = 0,$$

which furnishes an equation of degree $n-3$ in λ.

13. The highest singular line of the second order and first species which a quartic surface can possess is a *tacnodal* line; but when the line is of the second species, such a surface may possess a *rhamphoid cuspidal* and an *osculodal line*. The equations of the surface in the two respective cases may be reduced to the forms

$$(\alpha\gamma + \delta^2)^2 + \gamma(\alpha\gamma v_1 + \beta\gamma w_1 + w_2) = 0 \ldots\ldots\ldots (16),$$

and

$$(\alpha\gamma + \delta^2)^3 + \gamma^3(P\alpha + Q\beta + R\gamma + S\delta) = 0 \ldots\ldots\ldots (17).$$

The section of (17) by the plane $\alpha = \lambda\beta$ is

$$(\lambda\beta\gamma + \delta^2)^2 + \gamma^3\{(P\lambda + Q)\beta + R\gamma + S\delta\} = 0,$$

and the condition that B' should be a tacnode cusp is that $\lambda = -Q/P$. An osculodal line of the second kind on a quartic has therefore one tacnode cuspidal point on it. It is not possible for a quartic to have a tacnode cuspidal line, since the conditions are that $P\lambda + Q = 0$ for all values of λ, which requires that $P = Q = 0$, in which case (17) becomes a cone.

Triple Lines.

14. The tangent planes at any point of a multiple line may be fixed or torsal; also two or more tangent planes may coincide along the whole length of the line. We thus obtain ten *primary* species of triple lines.

I. Three distinct tangent planes; all of which are torsal.

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* *Trinodal and Quadrinodal Quintics*, vol. XXXVI., p. 109, equation (9).

II. Three distinct tangent planes; one fixed and two torsal.

III. Three distinct tangent planes; two fixed and one torsal.

IV. Three distinct tangent planes; all three fixed.

V. Two coincident fixed tangent planes; one distinct torsal plane.

VI. Two coincident fixed tangent planes; one distinct fixed plane.

VII. Three coincident fixed tangent planes.

VII. Two coincident torsal tangent planes; one distinct torsal plane.

IX. Two coincident torsal tangent planes; one distinct fixed plane.

X. Three coincident torsal tangent planes.

Triple lines possess a variety of species of singular points, which we shall proceed to consider. Thus when the line is of the first kind, points exist at which a pair of tangent planes coincide, so that the section of the surface through the point has a triple point of the second species thereat; also in certain cases points exist which are quartic nodes on the triple line.

I. *A triple line of the first kind on a surface of the n^{th} degree has $4n-12$ points at which two of the tangent planes coincide; and $7n-24$ points, such that any section through one of them has a triple point thereat, one of whose tangents osculates its own branch.*

The equation of the surface is of the form

$$(P, Q, R, S\mathfrak{X}\gamma, \delta)^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (18),$$

where P, Q, R, S are quarternary quantics of degree $n-3$. Equation (18), when written out at full length, becomes

$$\alpha^{n-3}v_3 + \alpha^{n-4}(\beta w_3 + w_4) + \ldots \beta^{n-3}W_3 + \beta^{n-4}W_4 + \ldots W_n = 0 \ldots (19),$$

and the equation of the section by the plane $\alpha = \lambda\beta$ is

$$\beta^{n-3}(\lambda^{n-3}v_3 + \lambda^{n-4}w_3 + \ldots W_3) + \beta^{n-4}(\lambda^{n-4}w_4 + \lambda^{n-5}V_4 + \ldots W_4) + \ldots = 0 \ldots (20),$$

or

$$A\beta^{n-3} + B\beta^{n-4} + \ldots = 0 \ldots\ldots\ldots\ldots (21).$$

The points at which a pair of tangent planes coincide will be called pinch points, and the condition for their existence is

that the discriminant of A should vanish; and since A is a binary cubic of (γ, δ) whose coefficients are polynomials of λ of degree $n-3$, the discriminant is of degree $4n-12$ in λ.

If one of the tangents osculates its own branch, A and B must have a common factor; and since the eliminant is of degree $4(n-3)+3(n-4)=7n-24$ in λ, there are $7n-24$ of such points.

Every tangent plane touches the triple line at $n-3$ distinct points.

The condition that γ should be a tangent plane at the point B' where the plane $\alpha=\lambda\beta$ cuts AB, is that the coefficient of δ^3 in A should vanish. This furnishes an equation of degree $n-3$ in λ, which shows that there are $n-3$ of such points.

II. *When $2n-6$ pinch points coincide in pairs, one of the tangent planes is fixed in space, and the line becomes one of the second species.*

Let γ be the fixed tangent plane, then A must be of the form

$$A=\gamma(\lambda^{n-3}v_2+\lambda^{n-4}w_2+\dots W_2)=\gamma A_1 \text{ (say)}.$$

Now pinch points may arise in two ways. In the first place when the coefficient of δ in A_1 vanishes, which gives $n-3$ pinch points; and secondly when the discriminant of A_1 vanishes, which furnishes $2n-6$ more. Hence the total number of apparent pinch points is $3n-9$, showing that $2n-6$ coincide in pairs.

III. *When all the pinch points coincide in pairs, two tangent planes are fixed in space.*

In this case

$$A=\gamma\delta(\lambda^{n-3}v_1+\dots W_1)=\gamma\delta A_1'.$$

Pinch points occur when the coefficient of γ or δ in A_1' vanishes; hence there are altogether $2n-6$ apparent pinch points.

IV. *When all the tangent planes are fixed in space, the triple line possesses $n-3$ quartic nodes.*

In this case

$$A=\gamma\delta v_1(\lambda^{n-3}v_0+\dots W_0)=\gamma\delta v_1 A_0,$$

and a quartic node will occur whenever A_0 vanishes. It will hereafter be shown that the pinch points coincide in quartettes at each quartic nodes.

The equation of the surface may be written in the form

$$\gamma\delta v_1(\alpha, \beta)^{n-3}+(P, Q, R, S, T\chi\gamma, \delta)^4=0,$$

and the points where quartic nodes occur are given by the equation $(\alpha, \beta)^{n-3}=0$; hence if A is a quartic node, the nodal cone is of the form

$$\beta\gamma\delta v_1 + v_4 = 0,$$

which is the equation of a quartic cone having a triple generator of the first kind.

V. *When all the pinch points coincide in quartettes, two coincident tangent planes are fixed in space, and the third one is torsal; also the line possesses $n-4$ singular points, at which the triple point of the second kind changes to one possessing a pair of tacnodal branches and one distinct ordinary branch.*

The value of A is of the form

$$A = \gamma^2 B_1,$$

and the condition for a pinch point is that the coefficient of δ in B_1 should vanish, which shows that there are $n-3$ apparent pinch points.

Equation (21) now becomes

$$B_1\beta^{n-3}\gamma^2 + B\beta^{n-4} + \ldots = 0,$$

and if the coefficient of δ in B vanishes, γ will be a factor of B, and the point consists of a pair of tacnodal branches and one ordinary branch through it. Since B is of degree $n-4$ in λ, there are $n-4$ of such points, and, like pinch points, they affect the class of the surface. A third species of singular points occurs when B and B_1, which is a linear function of (γ, δ), have a common factor, in which case the tangent to the ordinary branch osculates it; but as these points do not affect the class of the surface, they are comparatively unimportant.

VI. *When two coincident tangent planes are fixed in space, and the distinct plane is also fixed, the triple line possesses $n-3$ quartic nodes, and $n-4$ of the points considered in* V.

For the equation of the section by the plane $\alpha = \lambda\beta$ is

$$(\lambda^{n-3}v_0 + \ldots W_0)\beta^{n-3}\gamma^2\delta + B\beta^{n-4} + \ldots = 0.$$

VII. *When three coincident tangent planes are fixed in space, the line possesses $n-3$ quartic nodes, and $n-4$ points consisting of a pair of tacnodal branches and a coincident ordinary branch**.

The equation of the section is

$$A_0\beta^{n-3}\gamma^3 + B\beta^{n-4} + \ldots = 0.$$

* The theory of these points is given in S.C., §§ 3 to 6.

The first kind of points occur when $A_0 = 0$, and the second when γ is a factor of B. The constituents of the latter point (on a plane curve) are three nodes and one cusp; and both kinds of points affect the class of the surface.

15. Before discussing the remaining three species of triple lines, I shall explain another method for investigating the theory of pinch points, which will furnish a more rigorous proof of the coincidence of such points. Let Δ be the discriminant of (18), so that

$$\Delta = P^2S^2 - 6PQRS + 4PR^3 + 4Q^3S - 3Q^2R^2 \ldots (21A),$$

then $\Delta = 0$ is a surface of degree $4n-12$, and we shall first show that the pinch points occur where AB intersects the surface $\Delta = 0$. Let

$$P = P_0\alpha^{n-3} + P_1\alpha^{n-4} + \ldots P_{n-3},$$

with similar expressions for Q, R, S, where $P_{n-3} = (\beta, \gamma, \delta)^{n-3}$; then if A be a pinch point and $\gamma^2\delta = 0$ the equation of the tangent planes thereat, it follows from (18) that

$$P_0 = R_0 = S_0 = 0.$$

The term $4Q^3S$ in $(21A)$ contains the highest power of α which is the $(4n-13)^{\text{th}}$, and shows that the surface $\Delta = 0$ passes through A.

In the next plane consider the line II. in which $\gamma = 0$ is the fixed tangent plane. The values of P, Q, and R remain unaltered, but $S = \Sigma\gamma + T\delta$, where Σ, $T \equiv (\alpha, \beta, \gamma, \delta)^{n-4}$. Let

$$\Sigma = \sigma_0\alpha^{n-4} + \sigma_1\alpha^{n-5} + \ldots \sigma_{n-4},$$

$$T = t_0\alpha^{n-4} + t_1\alpha^{n-5} + \ldots t_{n-4},$$

where σ_n, $t_n \equiv (\beta, \gamma, \delta)^n$; then the two terms in $(21A)$ which contain the highest powers of α are

$$4Q_0^3(\gamma\sigma_0 + \delta t_0)\alpha^{4n-13} - 3Q_0^2R_1^2\alpha^{4n-14},$$

putting $\gamma = \delta = 0$, this reduces to $k\beta^2\alpha^{4n-14}$, where k is a constant, which shows that the line AB touches the surface $\Delta = 0$ at A, and therefore two pinch points coincide. If however we had supposed that the tangent planes at A were $\gamma\delta^2$, we should find that $\Delta = 0$ intersects but does not touch AB at A, so that A is an ordinary pinch point. Accordingly the discriminantal surface cuts AB in $2n-6$ points and touches it at $n-3$ points, which shows that there are $2n-6$ distinct pinch points and $2n-6$ which coincide in pairs.

In the same way any other case may be treated.

We shall now inquire what becomes of the pinch points in the case of line IV. Let M be a binary quantic of (α, β) of degree $n-3$; also let P, P', &c. be quaternary quantics of degree $n-4$. Then the equation of the surface may be written in the form

$$\gamma^3(P\gamma + Q\delta) + 3\gamma^2\delta\,(Mf + P'\gamma + Q'\delta) + 3\gamma\delta^2(Mg + R'\gamma + S'\delta) + \delta^3(R\gamma + S\delta) = 0,$$

where $\gamma\delta\,(f\gamma + g\delta) = 0$ are the three fixed tangent planes. Writing down the equation of the discriminantal surface, and then putting $\gamma = \delta = 0$, it will be found to reduce to $-3f^2g^2M^4 = 0$, which shows that the surface Δ has quadritactic contact with the line AB at the quartic nodes. This shows that the pinch points coincide in quartettes at the quartic nodes.

16. VIII. The remaining three species present many features in common with cuspidal lines of the first kind; and the equation of a surface having a line of the eighth species is

$$(L\alpha\gamma + M\beta\delta)^2(F, G, \ldots\chi\alpha, \beta)^{n-5} + (P, Q, R, S, T\chi\gamma, \delta)^4 = 0 \ldots\ldots\ldots\ldots(22),$$

where F, G, ... are linear functions of (γ, δ), and P, Q, ... are quaternary quantics of $(\alpha, \beta, \gamma, \delta)$. Equation (22) when written out at full length is of the form

$$(L\alpha\gamma + M\beta\delta)^2(F\alpha^{n-5} + G\alpha^{n-6}\beta + \ldots K\beta^{n-5}) + \alpha^{n-4}v_4 + \alpha^{n-5}(\beta V_4 + V_5) + \ldots + \beta^{n-4}w_4 + \ldots w_n = 0 \ldots(23),$$

Equation (23) shows that no surface of a lower degree than a sextic can possess a line of this species; for if $n = 5$, (23) reduces to the form

$$(L\alpha\gamma + M\beta\delta)^2v_1 + av_4 + \beta w_4 + w_5 = 0 \ldots\ldots(24),$$

in which the distinct tangent plane is fixed in space, and the line therefore belongs to species IX.

The section of (23) by the plane $\alpha = \lambda\beta$ is

$$(L\lambda\gamma + M\delta)^2(F\lambda^{n-5} + G\lambda^{n-6} + \ldots)\beta^{n-3} + (\lambda^{n-4}v_4 + \lambda^{n-5}V_4 + \ldots)\beta^{n-4} + \ldots = 0,$$

which we shall write

$$Av_1^2\beta^{n-3} + B\beta^{n-4} + \ldots = 0 \ldots\ldots\ldots\ldots(25),$$

(*a*) *The first kind of singular point occurs when all the tangent planes coincide, and there are* $n-4$ *of them.*

The condition for these points is that $A = kv_1$, where k is a constant, which furnishes an equation of degree $n-4$ in λ.

(*b*) *There are n points at which the triple point of the second kind changes into one consisting of one pair of tacnodal branches and an ordinary branch passing through it.*

The condition for these points is that v_1 should be a factor of B, which furnishes an equation of degree n in λ.

(*c*) *There are* $5n-24$ *points where the tangent to the ordinary branch osculates it.*

The condition is that the eliminant of A and B should vanish, which furnishes an equation of degree

$$n-4+4(n-5)=5n-24 \text{ in } \lambda.$$

The points (*a*) and (*b*) alone affect the class of the surface.

IX. When the distinct tangent plane is fixed in space, the equation of the surface is

$$(L\alpha\gamma + M\beta\delta)^2(p\gamma+q\delta)(\alpha,\beta)^{n-5} + (P, Q, R, S, T\chi\gamma, \delta)^4 = 0 \ldots\ldots (26),$$

and the section by the plane $\alpha = \lambda\beta$ is

$$(L\lambda\gamma + M\delta)^2(p\gamma+q\delta)(F\lambda^{n-5}+\ldots)\beta^{n-3} + (\lambda^{n-4}v_4 + \lambda^{n-5}V_4 + \ldots)\beta^{n-4} + \ldots = 0,$$

or

$$Av_1^2w_1\beta^{n-3} + B\beta^{n-4} + \ldots = 0 \ldots\ldots\ldots\ldots (27),$$

(*a*) *There are* $n-5$ *quartic nodes which are the intersections of the triple line and the planes* $(\alpha,\beta)^{n-5}=0$.

A quintic surface cannot possess these quartic nodes, and therefore the singularity occurs on such a surface in an incomplete form. When $n>5$ the equation of the nodal cone is of the form

$$\gamma^2(p\gamma+q\delta)\beta + v_4 = 0.$$

(*b*) *There is one pinch point, which occurs when* $Lq\lambda = Mp$.

(*c*) *There are n points where the triple point of the second kind changes into one consisting of a pair of tacnodal branches and one distinct ordinary branch.*

(*d*) *There are* $n-4$ *points where the tangent to the ordinary branch osculates it.*

These are found by equating the eliminant of w_1 and B to zero.

The points (*a*), (*b*), and (*c*) affect the class of the surface.

X. The equation of the surface is

$$(L\alpha\gamma + M\beta\delta)^3(\alpha, \beta)^{n-6} + (P, Q, R, S, T)(\gamma, \delta)^4 = 0 \ldots (28),$$

and the section by the plane $\alpha = \lambda\beta$ is

$$(L\lambda\gamma + M\delta)^3(F\lambda^{n-6} + \ldots)\beta^{n-3} + (\lambda^{n-4}v_4 + \ldots)\beta^{n-4} + \ldots = 0,$$

or

$$A\beta^{n-3}v_1^3 + B\beta^{n-4} + \ldots = 0.$$

(*a*) *There are* $n-6$ *quartic nodes, and the equation of the nodal cone is of the form*

$$\gamma^3\beta + v_4 = 0.$$

A sextic is the surface of lowest degree which can possess this line, and since there are no quartic nodes the singularity occurs in an incomplete form.

(*b*) *There are n points at which the triple point of the third kind changes into one consisting of a pair of tacnodal branches and one coincident ordinary branch.*

Both these singular points affect the class of the surface.

Nodal Curves.

17. The equation of a surface of the n^{th} degree which has a plane nodal curve of degree s is

$$\alpha^2 V_{n-2} + 2\alpha\Omega_s u_{n-s-1} + \Omega_s^2 u_{n-2s} = 0 \ldots\ldots\ldots\ldots (29)$$

where V is a quarternary quantic of $(\alpha, \beta, \gamma, \delta)$, and Ω, u are ternary quantics of (β, γ, δ) of the degrees indicated by the suffixes. We shall usually omit the suffix s in Ω.

18. *The nodal curve possesses* $2s(n-s-1)$ *pinch points, which are the points of intersection of the curve and surface*

$$u^2_{n-s-1} = V_{n-2}u_{n-2s} \ldots\ldots\ldots\ldots\ldots (30).$$

Let B be one of the points of intersection of (30) and the nodal curve, and let u_{n-2} be the portion of V_{n-2} which is independent of α; then (30) may be replaced by

$$u^2_{n-s-1} = u_{n-2}u_{n-2s} \ldots\ldots\ldots\ldots\ldots (31).$$

Since $\delta = 0$ is any arbitrary section of the surface through B, we have to show that the section has a cusp at B; also since (31) has to pass through B which is a point on Ω, it follows that when $\delta = 0$

$$\left.\begin{aligned} u_{n-2} &= p^2\beta^{n-2} + p_1\beta^{n-3}\gamma + \ldots p_{n-2}\gamma^{n-2} \\ u_{n-s-1} &= pq\beta^{n-s-1} + q_1\beta^{n-s-2}\gamma + \ldots \\ u_{n-2s} &= q^2\beta^{n-2s} + r_1\beta^{n-2s-1}\gamma + \ldots \\ \Omega_s &= k_1\beta^{s-1}\gamma + k_2\beta^{s-2}\gamma^2 + \ldots \end{aligned}\right\} \ldots\ldots\ldots (32).$$

From these results it follows that the highest power of β in the section of (29) by δ is the $(n-2)^{\text{th}}$ and that its coefficient is $(p\alpha+qk_1\gamma)^2$, which shows that B is a pinch point.

19. The plane α intersects the surface in the nodal curve twice repeated, and in the curve $\alpha=0$, $u_{n-2s}=0$, which is called the *residual curve*; and the latter curve intersects the nodal curve in $s(n-2s)$ points. When the curve is of a higher order of singularity, these points are as a general rule multiple points on the singular curve, but when the latter is nodal the plane α is a tangent plane at these points; in other words, α is one of the two nodal tangent planes. To prove this, let B be one of the points in question; then

$$V_{n-2}=\beta^{n-2}u_0'+\beta^{n-3}u_1'+\ldots,$$
$$u_{n-s-1}=\beta^{n-s-1}v_0+\ldots,$$
$$u_{n-2s}=\beta^{n-2s-1}w_1+\ldots,$$
$$\Omega=\beta^{s-1}\Omega_1+\ldots,$$

where $u_n'=(\alpha,\gamma,\delta)^n$, and $v_n\equiv w_n\equiv\Omega_n\equiv(\gamma,\delta)^n$, from which it follows that the coefficient of β^{n-2} in (29) is $\alpha(\alpha u_0'+2v_0\Omega_1)$.

Nodal curves also possess flecnodal points; but these do not affect the class of the surface and will not be considered.

Cuspidal Curves.

20. When the singular curve is cuspidal, every point on it must be a pinch point, which requires that equation (30) should contain Ω as a factor. Accordingly

$$u^2_{n-s-1}-u_{n-2}u_{n-2s}=\Omega\Omega' \quad \ldots\ldots\ldots\ldots\ldots\ldots(33),$$

where $\Omega'=(\beta,\gamma,\delta)^{2n-3s-2}$. The right-hand side of (33) vanishes at every point on the cuspidal curve, hence the left-hand side of (33) must do so also. Now u_{n-2s} vanishes at the points where the residual curve intersects the cuspidal curve, hence u_{n-s-1} must also vanish at these points; accordingly

$$u_{n\ s-1}=u_{n-2s}\sigma_{s-1}+\Omega w_{n-2s-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots(34),$$

where σ and w are undetermined ternary quantics of (β,γ,δ). Substituting from (34) in (33), we obtain

$$u^2_{n-2s}\sigma^2_{s-1}+2\Omega u_{n-2s}\sigma_{s-1}w_{n-2s-1}+\Omega^2w^2_{n-2s-1}-u_{n-2}u_{n\ 2s}=\Omega\Omega',$$

and since Ω has to be a factor of the left-hand side, we must have

$$u_{n-2}=u_{n-2s}\sigma^2_{s-1}+\Omega\phi_{n-s-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(35),$$

where ϕ is another ternary quantic of $(\beta,\gamma,\delta,)$.

Substituting from (34) and (35) in (29) and writing

$$S = \alpha\sigma_{s-1} + \Omega \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(36),$$

we obtain

$$S^2(u_{n-2s} + 2\alpha w_{n-2s-1}) + S\alpha^2(\phi_{n-s-2} - 4\sigma_{s-1}w_{n-2s-1}) + \alpha^3 U'_{n-3} = 0,$$

which is the equation of a surface having a plane cuspidal curve of degree s; but it can be easily verified that such a surface may be represented by the more general equation

$$S^2 V_{n-2s} + 2\alpha^2 S V_{n-s-2} + \alpha^3 W_{n-3} = 0 \ldots\ldots\ldots(37),$$

where U, V, W are quarternary quantics of the degrees indicated by the suffixes.

Some writers seem to have thought it possible that a surface possessing a cuspidal curve, plane or twisted, might be represented by an equation of the form $S^2 V + \Sigma^3 V = 0$, where S and Σ are two surfaces whose intersections determine the cuspidal curve; but although the above equation undoubtedly does represent such a surface, equation (37) indicates that it is deficient in generality, since it does not appear possible to transform (37) so as to get rid of the second term. Omitting suffixes, (37) may be written in the form

$$(SU + \alpha^2 V)^2 + \alpha^3(UW - V^2\alpha) = 0 \ldots\ldots\ldots(38),$$

which shows that the surface $UW = V^2\alpha$ touches (37) along its curve of intersection with the surface $SV + \alpha^2 V = 0$.

21. *The surface possesses $s(n-3)$ tacnodal points, which are the intersections of the surface $W_{n-3} = 0$ and the cuspidal curve.*

Let B be one of the points of intersection; then

$$S = \beta^{s-1}(\Omega_1 + \alpha\sigma_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(39),$$

$$W_{n-3} = \beta^{n-4} w_1 + \ldots,$$

and the coefficient of β^{n-2} in (37) is $(\Omega_1 + \alpha\sigma_0)^2$, and that of β^{n-3} contains $\Omega_1 + \alpha\sigma_0$ as a factor, which shows that B is a tacnodal point.

22. *The surface possesses $s(n-2s)$ cubic nodes, which are the points of intersection of the cuspidal and residual curves; also the nodal cone possesses a cuspidal generator, which is the tangent to the cuspidal curve at the point.*

Let B be one of the points of intersection of the two curves; then S is given by (39), whilst the value of U_{n-2s}, whose intersection with the plane α is the residual curve, is

$$U_{n-2s} = \beta^{n-2s-1} u_1 + \ldots,$$

where $u_n=(\alpha, \gamma, \delta)^n$. The highest power of β in (37) is the $(n-3)^{\text{th}}$, and its coefficient equated to zero is the cuspidal cubic cone

$$(\alpha\sigma_0+\Omega_1)^2 u_1+2\alpha^2(\alpha\sigma_0+\Omega_1)\, v_0+\alpha^3 w_0=0,$$

which is the cone in question.

Tacnodal Curves.

23. We have shown in § 21 that the tacnodal points on a cuspidal curve are its points of intersection with the surface $W_{n-3}=0$; hence if the curve is tacnodal, W_{n-3} must contain α as a factor, and the equation of a surface having such a curve is of the form

$$S^2{}_n U_{n-2s}+2\alpha^2 S V_{n-s-2}+\alpha^4 W_{n-4}=0 \ldots\ldots\ldots (40).$$

24. *There are $2s(n-s-2)$ points where the tacnode changes into a rhamphoid cusp; and they are the points of intersection of the tacnodal curve with the surface,*

$$V^2{}_{n-s-2}=U_{n-2s} W_{n-4} \ldots\ldots\ldots\ldots\ldots (41).$$

Let B be one of these points, then we must have

$$U_{n-2s}=p^2\beta^{n-2s}+\beta^{n-2s-1}u_1+\ldots u_{n-2s},$$
$$V_{n-s-2}=pq\beta^{n-s-2}+\ldots v_{n-s-2},$$
$$W_{n-4}=q^2\beta^{n-4}+\ldots w_{n-4},$$

where $u_n\equiv v_n\equiv w_n\equiv(\alpha, \gamma, \delta)^n$. Equation (40) may now be written

$$\beta^{n-2s}(pS+q\alpha^s\beta^{s-2})^2+S^2U'_{n-2s}+2\alpha^3 SV'_{n-s-2}+\alpha^4 W'_{n-4}=0 \ldots (42),$$

where U', V', W' are the portions of U, V, W which do not contain the highest power of β. Writing $\omega_1=\alpha\sigma_0+\Omega_1$, $\omega_n=(\alpha, \beta, \delta)^n$, (42) can be expressed in the form

$$\beta^{n-4}(p\beta\omega_1+p\omega_2+q\alpha^2)^2+\beta^{n-3}\omega_1{}^2\omega_1'+\beta^{n-4}\omega_1\omega_3+\beta^{n-5}\omega_5+\ldots=0,$$

which shows that B is a rhamphoid cusp on any section passing through it.

25. *The $s(n-2s)$ points where the tacnodal curve intersects the residual curve are cubic nodes, whose nodal cone consists of a quadric cone and a plane.*

If B be one of the points in question, the highest power of β is the $(n-3)^{\text{th}}$, and its coefficient equated to zero is

$$(\alpha\sigma_0+\Omega_1)\{(\alpha\sigma_0+\Omega_1)u_1+2\alpha^2 v_0\}=0,$$

which is the cone in question.

Rhamphoid Cuspidal Curves.

26. By pursuing a method similar to that employed in the case of a cuspidal curve, it can be shown that the equation of a surface possessing a rhamphoid cuspidal curve is

$$S^2 U_{n-2s} + 2\alpha^3 S V_{n-s-3} + \alpha^5 W_{n-5} = 0 \ldots\ldots\ldots (43),$$

where

$$S = \alpha^2 \sigma_{s-2} + \alpha \sigma_{s-1} + \Omega \ldots\ldots\ldots\ldots\ldots (44).$$

When $n=5$, $s=2$ equation (43) fails, and the correct equation for a quintic surface which possesses a rhamphoid cuspidal conic will be found by a different method later on.

27. *The $s(n-2s)$ points, where the residual curve intersects the singular curve, are cubic nodes of the fifth species.*

If B be one of these points, the highest power of β is the $(n-3)^{\text{th}}$, and its coefficient equated to zero is

$$(\alpha\sigma_0 + \Omega_1)^3 u_1 = 0,$$

which consists of two coincident and one distinct plane.

28. *There are $s(n-5)$ osculodal points, which are the points of intersection of the residual curve and the rhamphoid cuspidal one.*

The value of S may be written in the form

$$\beta^{s-1}(\alpha\sigma_0' + \Omega_1) + \beta^{s-2}(\alpha^2\sigma_0' + \alpha\sigma_2' + \Omega_2) + \ldots = \beta^{s-1}\omega_1 + \beta^{s-2}\omega_2 + \ldots \quad \text{(say)},$$

where $\omega_n = (\alpha, \gamma, \delta)^n$. Putting $\delta = 0$, it can easily be shown that the conic $\beta\omega_1 + \omega_2 = 0$ has sextactic contact with the section at B, and tritactic contact with the section of the first polar by the plane $\delta = 0$; hence B is an osculode.

29. The equation of a quintic surface having a tacnodal conic is

$$S^2 U + 2SV\alpha^2 + W\alpha^4 = 0 \ldots\ldots\ldots\ldots (45),$$

where U, V, W are planes and

$$S = \alpha u + \Omega \ldots\ldots\ldots\ldots (46),$$

where u is another plane. The rhamphoid cuspidal points are the intersections of the plane α with the quadric cone $V^2 = UW$ hence if U_1, V_1, W_1 are the portions of these planes which are independent of α, it follows that when the conic is rhamphoid cuspidal

$$\Omega = k(V_1^2 - U_1 W_1) \ldots\ldots\ldots\ldots (47).$$

Equations (45), (46), and (47) determine a quintic surface having a rhamphoid cuspidal conic.

Singular Twisted Quartic Curves.

30. The theory of singular twisted curves appears to be of the same character. The equation

$$S^2U + 2S\Sigma V + \Sigma^2 W = 0 \ldots\ldots\ldots\ldots (48),$$

where S, Σ are quadric surfaces, and U, V, W quarternary quantics of degree $n-4$, represents a surface of the n^{th} degree having a nodal twisted quartic curve which is the intersection of S and Σ. The surface has $8(n-4)$ pinch points*, which are the intersections of the nodal curve and the surface $V^2 = UW$; for if B be one of these points of intersection, we must have

$$U = p^2\beta^{n-4} + \beta^{n-5}\, U_1 + \ldots,$$
$$V = pq\beta^{n-4} + \beta^{n-5}\, V_1 + \ldots,$$
$$W = q^2\beta^{n-4} + \beta^{n-5} W_1 + \ldots,$$
$$S = \beta v_1 + v_2, \quad \Sigma = \beta w_1 + w_2,$$

where the suffixed letters are ternary quantics of (α, γ, δ). From these equations it follows that the coefficient of β^{n-2} in (48) is $(pv_1 + qw_1)^2$, which shows that B is a pinch point.

31. When the twisted quartic is cuspidal, the discriminantal surface must contain the quartic, which requires that

$$U^2 - VW = \Phi S + \Psi\Sigma \ldots\ldots\ldots\ldots (49),$$

where Φ and Ψ are quantics of degree $2n-10$. Now the quadric S cuts (48) in the residual curve $S=0$, $W=0$, and since S, Σ, and W vanish at the points where the residual curve cuts the singular quantic, it follows that

$$V = BW + \phi S + \psi\Sigma \ldots\ldots\ldots\ldots (50),$$

where B, ϕ, ψ are quarternary quantics. But the quadric Σ cuts (48) in a second residual curve $\Sigma = 0$, $U = 0$; and at the points where this curve cuts the singular quartic, S, Σ, and U vanish: if therefore the value of V from (50) be substituted in (49) the preceding condition requires that $B=0$. Hence (49) becomes

$$(\phi S + \Psi\Sigma)^2 - VW = \Phi S + \psi\Sigma \ldots\ldots\ldots\ldots (51),$$

and consequently the condition that the left-hand side of (51) should be of the required form is that V or W should be of the form $\theta S + \chi\Sigma$. Taking the latter hypothesis and substituting in (48) we obtain the equation of the surface in the

* This is a particular case of a more general theorem due to Salmon, *Camb. and Dublin Math. Jour.*, vol. II., p. 65; see also *Quart. Jour.*, vol. XXXVIII., p. 165.

form

$$S^2 U_{n-4} + 2S\Sigma^2 V_{n-6} + \Sigma^3 W_{n-6} = 0 \ldots\ldots\ldots\ldots (52).$$

32. We shall now show that:—

There are $4(n-6)$ *tacnodal points which are the intersections of the surface* $W_{n-6}=0$ *with the cuspidal quartic; and* $4(n-4)$ *cubic nodes, at which there is a cuspidal cubic cone, which are the intersections of the surface* $U_{n-4}=0$ *with the curve.*

(i) Let B be one of the points of intersection of W_{n-5} with the cuspidal quartic; then

$$W_{n-6} = \beta^{n-7} W_1' + \beta^{n-8} W_2' + \ldots,$$
$$V_{n-6} = \beta^{n-6} V_0' + \ldots,$$
$$U_{n-4} = \beta^{n-4} U_0' + \ldots.$$

Substituting these values in (52) it becomes

$$\beta^{n-2} v_1^2 V_0' + \beta^{n-3} v_1 (2 V_0' v_2 + v_1 V_1' + 2 w_1^2 V_0') + \ldots = 0,$$

which shows that B is a tacnodal point.

(ii) Let B be one of the points of intersection of U_{n-4} with the cuspidal quartic; then we must add the term $W_0 \beta^{n-6}$ to W_{n-6}, and expunge the term $\beta^{n-4} U_0'$ from U_{n-4}, and (52) becomes

$$\beta^{n-3} (v_1^2 U_1' + 2 v_1 w_1^2 V_0' + w_1^3 W_0') + \ldots = 0,$$

which shows that B is a cubic node, at which there is a cuspidal cubic tangent cone.

Singular Twisted Cubic Curves.

33. The theory of nodal curves, which are the partial intersections of two surfaces, has been discussed by Cayley*; but he has not given the number of pinch points or considered curves of a higher singularity. Salmon† has also, by a very obscure method, arrived at the conclusion that when the surface is a quartic, the number of pinch points is 4. Let u, u'; v, v'; w and w' be quarternary quantics of degrees l, m, and n respectively; then Salmon‡ has shown that the system of determinants

$$\left\| \begin{matrix} u, & v, & w \\ u', & v', & w' \end{matrix} \right\| = 0 \ldots\ldots\ldots\ldots\ldots (53)$$

* On a Singularity of Surfaces, *Collected Papers*, vol. VI., p. 123; and *Quart. Jour.*, vol. IX., p. 332.

† *Surfaces*, 3rd edition, p. 487.

‡ *Higher Algebra*, 4th edition, p. 286.

represents three surfaces having a common curve of intersection of degree $mn+nl+lm$. Hence if

$$\lambda = vw' - wv', \quad \mu = wu' - uw', \quad \nu = uv' - vu' \ldots (54),$$

the equation

$$(U, V, W, U', V', W'\chi\lambda, \mu, \nu)^2 = 0 \ldots\ldots (55),$$

where U, V, ... are quarternary quantics of proper degrees, represents a surface having a nodal twisted curve, which is the common curve of intersection of (53).

34. To apply this to a cubic curve, all the quantities u, u', &c. must be planes, and U, V, &c. must be quarternary quantics of degrees $n-4$. Also if A be a point on the cubic, the quadrics λ, μ, ν must pass through A, the condition for which may be satisfied by taking

$$u = \alpha u_0 + u_1, \; u_0' = k\alpha u_0 + u_1' \ldots\ldots\ldots\ldots (56),$$

with similar expressions for v, v' ; w, w'. Hence

$$\lambda = \alpha\{(kv_1 - v_1')w_0 - (kw_1 - w_1')v_0\} + v_1w_1' - w_1v_1'$$
$$= \alpha\lambda_1 + \lambda_2 \quad \text{(say)},$$

accordingly $\lambda_1 u_0 + \mu_1 v_0 + \nu_1 w_0 = 0 \ldots\ldots\ldots\ldots (57).$

Let $U = U_0\alpha^{n-4} + U_1\alpha^{n-5} + \ldots$

with similar expressions for V, W, &c.; then if these values be substituted in (55), the highest power of α will be α^{n-2}, and its coefficient equated to zero gives

$$(U_0, V_0, W_0, U_0', V_0', W_0'\chi\lambda_1, \mu_1, \nu_1)^2 = 0 \ldots\ldots (58),$$

which by virtue of (57) is resolvable into two linear factors. Hence (58) combined with (57) gives the nodal tangent planes at A.

To find the number of pinch points, consider the determinantal surface

$$\Delta = \begin{vmatrix} U, & W', & V', & u \\ W', & V, & U', & v \\ V', & U', & W, & w \\ u, & v, & w, & 0 \end{vmatrix} = 0 \ldots\ldots\ldots\ldots (59),$$

which is of degree $2n-6$, and the first term of which is $\Delta_0\alpha^{2n-6}$, where Δ_0 is the value of Δ, when U_0, V_0... are substituted for U, V.... .

If A be a pinch point (58) must become a perfect square, the condition for which is the same as the condition that the

line (57) should touch the conic (58), where λ_1, μ_1, ν_1 are regarded as ordinary trilinear coordinates of a point; that is to say $\Delta_0 = 0$. Hence the determinantal surface passes through all the pinch points, and there are apparently $6(n-3)$ of them. But the points of intersection of the three planes u, v, and w is a conic node on Δ, and this point can be shown to be an ordinary point on the cubic curve. For if the tetrahedron of reference be changed by writing (β, γ, δ) for (u, v, w), the coefficient of α^{n-2} in (55) becomes

$$(U_0, V_0 \ldots \gamma w_0' - \delta v_0', \delta u_0' - \beta w_0', \beta v_0' - \gamma u_0')^2,$$

which is not a perfect square. Hence the cubic curve intersects the surface Δ at a node and at $6n-20$ ordinary points, which are the number of pinch points on the nodal curve. When the surface is a quartic, $6n-20=4$, which agrees with Salmon's result.

35. These results are capable of generalization. For if the degrees of the coefficients in (55) are as follows:—

$$U = s-2m-2n\ ; \quad U' = s-2l-m-n\ ;$$
$$V = s-2n-2l\ ; \quad V' = s-l-2m-n\ ;$$
$$W = s-2l-2m\ ; \quad W' = s-l-m-2n\ ;$$

equation (55) will represent a surface of degree s having a nodal twisted curve of degree $mn+nl+lm$. In the place of (56), we must write

$$u = \alpha^l u_0 + \alpha^{l-1} u_1 + \ldots,$$
$$u' = k u_0 \alpha^l + \alpha^{l-1} u_1' + \ldots,$$

&c., and equation (57) will still hold good. Also, if

$$A = VW - U'^2, \quad A' = V'W' - UU', \quad \&c. \ldots . (59a),$$

the determinantal equation (59) becomes

$$(A, B, C, A', B', C' \mathbin{\text{X}} u, v, w)^2 = 0 \ldots\ldots (60),$$

which is of degree $2(s-l-m-n)$; and the degrees of A, A' are respectively equal to $2(s-2l-m-n)$ and $2s-2l-3m-3n$, so that there are apparently $2(mn+nl+lm)(s-l-m-n)$ pinch points. But the three points of intersection of the surfaces u, v, and w are nodes on (60) and are ordinary points on the nodal curve; hence the total number of pinch points is

$$2\{s(mu+nl+lm) - l^2(m+n) - m^2(n+l) - n^2(l+m) - 4lmn\}.$$

These results hold good whenever it is possible to represent a surface having a nodal twisted curve by means of an equation such as (55).

36. On a future occasion I hope to discuss the question of singular twisted curves on surfaces, but it may be worth while to show that if a quartic surface possesses a cuspidal twisted cubic, it is a developable surface whose edge of regression* is the cubic.

The equation of the twisted cubic may be expressed by the equations

$$u/v = v/w = w/t \ldots\ldots\ldots\ldots\ldots\ldots (61),$$

where u, v, w, t are planes; whence

$$\lambda = vt - w^2, \quad \mu = wv - ut, \quad \nu = uw - v^2 \ldots\ldots (62),$$

and the determinantal quadric (59) may be written in the form

$$Au^2 + (B + 2B')uw + Cvt + 2A'ut + 2C'uv - B\nu - C\lambda + 2A'\mu = 0 \ldots\ldots\ldots\ldots (63),$$

where A, B, ... are given by (59a).

Now if the quartic possesses a cuspidal cubic curve, every point on it must be a pinch point, and the surface (63) must contain the curve. This requires that

$$\left.\begin{aligned} A = C = A' = C' = 0 \\ B + 2B' = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (64),$$

which lead to the equations $U = U' = W = W' = 0$, and give $B = -V'^2$, $B' = -VV'$, so that $V' = -2V$. The equation of the quartic now becomes

$$(wv - ut)^2 = 4(uw - v^2)(vt - w^2) \ldots\ldots\ldots\ldots (65),$$

which is the discriminant of the equation

$$u\theta^3 + 3v\theta^2 + 3w\theta + t = 0 \ldots\ldots\ldots\ldots\ldots\ldots (66),$$

The quartic (65) is therefore the envelope of the plane (66), where θ is a variable parameter, and is therefore a developable surface whose edge of regression is (61). If A is any point on the cubic, we may write

$$u = \alpha + u_1, \quad v = \alpha + v_1,$$
$$w = \alpha + w_1, \quad t = \alpha + t_1$$

and the coefficient of α^2 in (65) is $(u_1 - 3v_1 + 3w_1 - t_1)^2$, which gives the cuspidal tangent plane at A.

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* The Italians call an ordinary cusp *un regresso;* a rhamphoid cusp *un regresso di seconda specie*, and so on.

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SOME MULTIPLE INTEGRALS.

By G. H. Hardy, Trinity College, Cambridge.

§ 1. IN this paper I propose to investigate the values of certain multiple integrals which contain Bessel functions under the sign of integration. I shall, however, be concerned less with the general formulæ than with a number of interesting special cases contained in them. Incidentally I shall evaluate a number of simple integrals—the chief of these will be found in § 3. The variety of integrals of this kind which have been evaluated by Weber, Sonine, Gegenbauer, Nielsen, and other writers is so great that I can hardly imagine this formula to be new: but I have never actually come across it or some of the deductions which I draw from it.

The function $\psi_\alpha(x)$.

§ 2. Let $\psi_\alpha(x)$ be the function defined for all real* values of α and all positive values of x by the formula

$$\psi_\alpha(x) = \int_0^\infty e^{-t^2-(x^2/t^2)}\, t^\alpha\, dt, \tag{1}$$

so that, in particular,

$$\psi_0(x) = \tfrac{1}{2}\sqrt{\pi}\, e^{-2x}. \tag{2}$$

It is easy to see that $\psi_\alpha(x)$ tends exponentially to zero as x tends to infinity, for any value of α.

By putting $x/t = u$, we obtain the formula

$$\psi_\alpha(x) = x^{\alpha+1}\psi_{-\alpha-2}(x). \tag{3}$$

We may also notice that

$$\psi_\alpha'(x) = -2x\,\psi_{\alpha-2}(x) = -2x^\alpha\psi_{-\alpha}(x). \tag{4}$$

If we put

$$t^2 = u,\ x^2 = \xi,\ \alpha = 2\beta - 1,$$

we see that

$$\psi_{2\beta-1}(\sqrt{\xi}) = \tfrac{1}{2}\int_0^\infty e^{-u-(\xi/u)}u^{\beta-1}du\,; \tag{5}$$

and by means of the formulæ†

$$\int_0^\infty e^{-u-(\xi/u)}u^{\beta-1}du = \frac{2^{2\beta+1}\xi^\beta e^{-2\sqrt{\xi}}\sqrt{\pi}}{\Gamma(\frac{1}{2}+\beta)}\int_0^\infty \frac{e^{-4t\sqrt{\xi}}}{\{t(1+t)\}^{\beta-\frac{1}{2}}}\,dt, \tag{6}$$

* The integral defines the function for all values of α, real or complex; but we shall be connected with real values only.

† (6) is given by Dirichlet (Meyer's edition of his lectures, p. 289), and (7) by Kummer (*Crelle's Journal*, bd. 17).

$$(7)\quad \int_0^\infty e^{-u-(\xi/u)}u^{\beta-1}du=\frac{2\Gamma(\frac{1}{2}+\beta)}{\sqrt{\pi}}\int_0^\infty\frac{\cos(2t\sqrt{\xi})}{(1+t^2)^{\beta+\frac{1}{2}}}dt,$$

we obtain the expressions in terms of this function of two other interesting integrals. The function may be expressed in terms of Bessel functions without difficulty. Thus Kummer* proves the formula

$$(8)\quad \int_0^\infty e^{-u-(\xi/u)}u^{\beta-1}du=\Gamma(\beta)\phi(1-\beta,\ \xi)+\Gamma(-\beta)\xi^\beta\phi(1+\beta,\ \xi),$$

where
$$\phi(s,\ \xi)=1+\frac{\xi}{1.s}+\frac{\xi^2}{1.2.s.s+1}+\dots,$$

and this formula is equivalent to Sonine's formula†

$$(9)\quad \int_0^\infty e^{-u-(\xi/u)}u^{\beta-1}du$$
$$=\frac{\pi\xi^{\frac{1}{2}\beta}}{\sin\beta\pi}\{i^\beta J_{-\beta}(2i\sqrt{\xi})-i^{-\beta}J_\beta(2i\sqrt{\xi})\}$$
$$=\pi\xi^{\frac{1}{2}\beta}e^{\frac{1}{2}(1+\beta)\pi i}H_1^{\ \beta}(2i\sqrt{\xi}),$$

where $H_1^{\ \nu}(x)$ is Hankel's cylinder function defined‡ by the equation

$$(10)\quad H_1^{\ \nu}(x)=(i/\sin\nu\pi)\{e^{-\nu\pi i}J_\nu(x)-J_{-\nu}(x)\}$$

All of these formulæ are, of course, subject to certain restrictions on the values of the constants involved. Thus (8) and (9) require modification when β is an integer: in the most interesting case, that in which $\beta=0$, we have

$$(11)\quad \int_0^\infty e^{-u-(\xi/u)}(du/u)=\pi\{iJ_0(2i\sqrt{\xi})-Y_0(2i\sqrt{\xi})\}.$$

The preceding formulæ also define the analytic continuation of $\psi_\alpha(x)$ all over the plane of x, but with this we are not at present concerned. We note that if $\alpha>-1$

$$(12)\quad \lim_{x\to 0}\psi_\alpha(x)=\tfrac{1}{2}\Gamma\{\tfrac{1}{2}(1+\alpha)\},$$

while if $\alpha<-1$

$$(13)\quad \psi_\alpha(x)\sim\tfrac{1}{2}\Gamma\{-\tfrac{1}{2}(1+\alpha)\}x^{1+\alpha},$$

as x approaches 0. Finally, if $\alpha=-1$,

$$(14)\quad \psi_{-1}(x)\sim-\log x.$$

A definite integral.

§ 3. Consider the formula

$$\psi_\alpha\sqrt{(c^2+x^2)}=\int_0^\infty e^{-t^2-(c^2+x^2)/t^2}t^\alpha dt.$$

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* *l. c.* † *Math. Ann.*, bd. 16, p. 44. ‡ Nielsen, *Handbuch*, p. 17.

Multiply by $x^{\nu+1}J_\nu(2\mu x) \quad (\mu > 0)$,

and integrate from $x=0$ to $x=\infty$. In virtue of the formula*

$$(15) \qquad \int_0^\infty e^{-x^2/t^2} x^{\nu+1} J_\nu(2\mu x)\, dx = \tfrac{1}{2}\mu^\nu t^{2\nu+2} e^{-\mu^2 t^2},$$

we obtain

$$(16) \quad \int_0^\infty \psi_\alpha\{\sqrt{(c^2+x^2)}\}\, x^{\nu+1} J_\nu(2\mu x)\, dx$$
$$= \tfrac{1}{2}\mu^\nu \int_0^\infty e^{-(1+\mu^2)t^2-(c^2/t^2)} t^{\alpha+2\nu+2}\, dt$$
$$= \tfrac{1}{2}\mu^\nu c^{\alpha+2\nu+3} \int_0^\infty e^{-u^2-\{c^2(1+\mu^2)/u^2\}} u^{-\alpha-2\nu-4}\, du$$

(on putting $c/t = u$)

$$= \tfrac{1}{2}\mu^\nu c^{\alpha+2\nu+3} \psi_{-\alpha-2\nu-4}\{c\sqrt{(1+\mu^2)}\},$$

except when $c=0$, in which case we obtain simply

$$(17) \qquad \int_0^\infty \psi_\alpha(x)\, x^{\nu+1} J_\nu(2\mu x)\, dx = \tfrac{1}{4}\mu^\nu \frac{\Gamma\{\frac{1}{2}(\alpha+3)+\nu\}}{(1+\mu^2)^{\frac{1}{2}(\alpha+3)+\nu}}.$$

So far, however, we have not considered in what circumstances our procedure is justifiable.

We have to justify the equation

$$\int_0^\infty x^{\nu+1} J_\nu(2\mu x)\, dx \int_0^\infty e^{-t^2-(c^2+x^2)/t^2} t^\alpha dt$$
$$= \int_0^\infty e^{-t^2-(c^2/t^2)} t^\alpha dt \int_0^\infty e^{-x^2/t^2} x^{\nu+1} J_\nu(2\mu x)\, dx.$$

Since, as x tends to 0,

$$x^{\nu+1} J_\nu(2\mu x) \sim \frac{\mu^\nu x^{2\nu+1}}{\nu!}$$

the inner integration with respect to x is possible if and only if $2\nu+1 > -1$ or $\nu > -1$. If this condition is satisfied, and $c>0$, both repeated integrals are convergent; and it is not difficult to see that the two are equal. For this will certainly be the the case if the double integral

$$\int_0^\infty \int_0^\infty e^{-t^2-(c^2+x^2)/t^2} t^\alpha x^{\nu+1} J_\nu(2\mu x)\, dt\, dx$$

is absolutely convergent. Now, if $0<x<1$, we have $|J_\nu(2\mu x)| < Kx^\nu$, and, if $x>1$, we have $|J_\nu(2\mu x)| < K/\sqrt{x}$, and it is easy to see that the integrals

$$\int_0^\infty \int_0^1 e^{-t^2-(c^2+x^2)/t^2} t^\alpha x^{2\nu+1} dt\, dx,$$
$$\int_0^\infty \int_1^\infty e^{-t^2-(c^2+x^2)/t^2} t^\alpha x^{\nu+\frac{1}{2}} dt\, dx$$

are convergent. Hence (16) holds if $\nu > -1$, *i.e.* whenever the integral has a meaning.

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* Nielsen, *l.c.*, p. 185.

It is almost equally easy to prove that (17) holds whenever the integral is convergent. The condition for this is

$$\nu > -1 \quad (\alpha \geqq -1),$$
$$2\nu + \alpha + 3 > 0 \quad (\alpha < -1).$$

In particular, if $\alpha = 0$, we obtain

$$(18)\quad \int_0^\infty e^{-2\sqrt{(c^2+x^2)}} x^{\nu+1} J_\nu(2\mu x)\, dx = \frac{\mu^\nu c^{2\nu+3}}{\sqrt{\pi}} \psi_{-2\nu-4}\{c\sqrt{(1+\mu^2)}\},$$

$$(19)\quad \int_0^\infty e^{-2x} x^{\nu+1} J_\nu(2\mu x)\, dx = \frac{\mu^\nu}{2\sqrt{\pi}} \frac{\Gamma(\nu+\frac{3}{2})}{(1+\mu^2)^{\nu+\frac{3}{2}}},$$

On writing ay for $2x$ and b for μa the last formula becomes

$$(20)\quad \int_0^\infty e^{-ay} y^{\nu+1} J_\nu(by)\, dy = \frac{2a\Gamma(\nu+\frac{3}{2})}{\sqrt{\pi}} \frac{(2b)^\nu}{(a^2+b^2)^{\nu+\frac{3}{2}}},$$

which is a well known formula*.

§ 4. There are a number of particular cases of the formulæ of § 3 which are interesting.

We may notice, in the first place, that $\psi_\alpha(x)$ may be expressed in finite terms whenever α is an even integer, positive or negative. For, differentiating the formula

$$\int_0^\infty e^{-t^2-(x^2/t^2)} dt = \tfrac{1}{2}\sqrt{(\pi)}\, e^{-2x}$$

n times with respect to x^2, we obtain

$$(21)\quad \int_0^\infty e^{-t^2-(x^2/t^2)} t^{-2n} dt = \tfrac{1}{2}\sqrt{\pi}\,\{-(d/dx^2)^n\, e^{-2x},$$

and, on putting $x/t = u$,

$$(22)\quad \int_0^\infty e^{-u^2-(x^2/u^2)} u^{2n-2} du = \tfrac{1}{2}\sqrt{(\pi)}\, x^{2n-1}\{-(d/dx^2)^n e^{-2x};$$

and these two formulæ give the value of $\psi_\alpha(x)$ whenever x is an even integer. The last formula agrees with the formula†

$$(23)\quad \int_0^\infty e^{-u^2-(x^2/u^2)} u^{2n-2} du$$
$$= \tfrac{1}{2}\sqrt{(\pi)} x^{n-1} e^{-2x}\left\{1 + \frac{n(n-1)}{2}\frac{1}{2x} + \frac{(n+1)n(n-1)(n-2)}{2.4}\left(\frac{1}{2x}\right)^2 + \ldots\right\}.$$

In particular

$$(24)\quad \psi_2(x) = \int_0^\infty e^{-u^2-(x^2/u^2)} u^2 du = \tfrac{1}{2}\sqrt{(\pi)}\, e^{-2x}(x+\tfrac{1}{2}).$$

It follows from what precedes that the integral (16) can be expressed in finite terms whenever $\alpha + 2\nu$ is an even integer,

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* Nielsen, *l.c.*, p. 186 † *Dirichlet-Meyer*, p. 289.

positive or negative. Consider, for example, the case in which $\alpha = \nu = 0$. We have then

$$(25)\quad \int_0^\infty e^{-2\sqrt{(c^2+x^2)}} x J_0(2\mu x)\, dx$$
$$= \frac{c^3}{\sqrt{\pi}} \psi_{-4}\{c\sqrt{(1+\mu^2)}\}$$
$$= \frac{e^{-2c\sqrt{(1+\mu^2)}}}{2(1+\mu^2)^{\frac{3}{2}}} \{\tfrac{1}{2} + c\sqrt{(1+\mu^2)}\},$$

by (3) and (24), a formula which for $c = 0$ reduces to

$$(26)\qquad \int_0^\infty e^{-2x} x J_0(2\mu x)\, dx = \tfrac{1}{4}(1+\mu^2)^{-\frac{3}{2}},$$

which is easily verified.

§ 5. If in the formulæ (16), (17) we put $\nu = -\frac{1}{2}$, so that

$$x^{\nu+1} J_\nu(2\mu x) = \frac{\cos 2\mu x}{\sqrt{(\mu\pi)}},$$

we obtain

$$(27)\quad \int_0^\infty \psi_\alpha\{\sqrt{(c^2+x^2)}\} \cos 2\mu x\, dx = \tfrac{1}{2}\sqrt{(\pi)} c^{\alpha+2} \psi_{-\alpha-3}\{c\sqrt{(1+\mu^2)}\}.$$

and

$$(28)\qquad \int_0^\infty \psi_\alpha(x) \cos 2\mu x\, dx = \tfrac{1}{4}\sqrt{\pi}\, \frac{\Gamma(\frac{1}{2}\alpha+1)}{(1+\mu^2)^{\frac{1}{2}\alpha+1}}.$$

the last of which may be verified immediately when $\alpha = 0$.

Other definite integrals which contain $\psi_\alpha(x)$.

§ 6. (i) If we multiply the first formula of § 3 by x^ν $(\nu > -1)$ and integrate from $x = 0$ to $x = \infty$, we obtain

$$(29)\quad \int_0^\infty \psi_\alpha \sqrt{(c^2+x^2)}\, x^\nu\, dx = \int_0^\infty e^{-t^2-(c^2/t^2)} t^\alpha dt \int_0^\infty e^{-(x^2/t^2)} x^\nu dx$$
$$= \tfrac{1}{2}\Gamma\{\tfrac{1}{2}(1+\nu)\} \int_0^\infty e^{-t^2-(c^2/t^2)} t^{\alpha+\nu+1} dt$$
$$= \tfrac{1}{2}\Gamma\{\tfrac{1}{2}(1+\nu)\}\, \psi_{\alpha+\nu+1}(c).$$

It is again easy to justify the inversion. If, however, $c = 0$, we obtain

$$(30)\quad \int_0^\infty \psi_\alpha(x)\, x^\nu dx = \tfrac{1}{4}\Gamma\{\tfrac{1}{2}(1+\nu)\}\, \Gamma\{1+\tfrac{1}{2}(\alpha+\nu)\}.$$

This formula holds if

$$\nu > -1 \quad (\alpha \geqq -1),$$
$$\alpha + \nu + 2 > 0 \quad (\alpha < -1),$$

and is substantially equivalent to Nielsen's formula*

* *l. c.*, p. .

(31) $\int_0^\infty H_1^{\nu}(tx\iota)\, t^{\rho}\, dt$

$$= \frac{2^{\rho} e^{-\frac{1}{2}(\nu+1)\pi\iota}}{\pi x^{1+\rho}} \Gamma\{\tfrac{1}{2}(\rho+\nu+1)\}\, \Gamma\{\tfrac{1}{2}(\rho-\nu+1)\}.$$

(ii) From the equation

$$\psi_a\{ax+(b/x)\} = \int_0^\infty e^{-t^2-\{ax+(b/x)\}^2/t^2} t^a dt,$$

where a and b are positive, we deduce

(32) $\int_0^\infty \psi_a\{ax+(b/x)\}dx = \int_0^\infty e^{-t^2-(2ab/t^2)} t^a dt \int_0^\infty e^{-(a^2x^2/t^2)-(b^2/t^2x^2)} dx$

$$= (\sqrt{\pi}/2a) \int_0^\infty e^{-t^2-(4ab/t^2)} t^{a+1} dt = (\sqrt{\pi}/2a)\, \psi_{a+1}(2\sqrt{ab}).$$

By means of the formula

$$\int_0^\infty f\{ax+(b/x)\}\, dx = (1/a) \int_0^\infty f\sqrt{(z^2+4ab)}\, dz,$$

this equation may be reduced to

(33) $$\int_0^\infty \psi_a \sqrt{(z^2+4ab)}\, dz = \tfrac{1}{2}\sqrt{(\pi)}\, \psi_{a+1}\{2\sqrt{(ab)}\},$$

which is a particular case of (27).

(iii) Consider the integral

$$\int_0^\infty \psi_a(ax)\, \psi_\beta(bx)\, x^{\nu} dx \quad (a, b > 0),$$

which is the same as

$$\int_0^\infty x^{\nu} dx \int_0^\infty e^{-t^2-(a^2x^2/t^2)} t^a dt \int_0^\infty e^{-u^2-(b^2x^2/u^2)} u^\beta du$$

$$= \int_0^\infty e^{-t^2} t^a dt \int_0^\infty e^{-u^2} u^\beta du \int_0^\infty e^{-\{(a^2/t^2)+(b^2/u^2)\}x^2} x^{\nu} dx.$$

In the last integral put $x = tu\xi$. We obtain

$$\int_0^\infty e^{-t^2} t^{\alpha+\nu+1} dt \int_0^\infty e^{-u^2} u^{\beta+\nu+1} du \int_0^\infty e^{-(b^2t^2+a^2u^2)x^2} x^{\nu} dx,$$

and on inverting the order of the integrations again, we arrive at the formula

(34) $\int_0^\infty \psi_a(ax)\, \psi_\beta(bx)\, x^{\nu} dx$

$$= \tfrac{1}{4}\Gamma\{1+\tfrac{1}{2}(\alpha+\nu)\}\Gamma\{1+\tfrac{1}{2}(\beta+\nu)\} \int_0^\infty \frac{x^{\nu} dx}{(1+a^2x^2)^{1+\frac{1}{2}(\alpha+\nu)}(1+b^2x^2)^{1+\frac{1}{2}(\beta+\nu)}}.$$

The integral on the right-hand side is convergent if

$$\nu > -1,\ \alpha+\beta+\nu > -3;$$

the integral on the left if

(i) $\alpha \geqq -1$, $\beta \geqq -1$, $\nu > -1$,

(ii) $\alpha < -1$, $\beta \geqq -1$, $\alpha+\nu > -2$,

(iii) $\alpha \geqq -1$, $\beta < -1$, $\beta+\nu > -2$,

(iv) $\alpha < -1$, $\beta < -1$, $\alpha+\beta+\nu > -3$;

and it is easy to prove (*a*) that the two sets of conditions are equivalent (*b*) that if they are satisfied the transformation is legitimate.

A number of particular cases included in this formula are of interest.

(*a*) Let $\alpha=\beta=0$. Then the integral on the left-hand side reduces to

$$\tfrac{1}{4}\pi\int_0^\infty e^{-2(a+b)x}x^\nu dx=\frac{\pi\Gamma(\nu+1)}{2^{\nu+3}(a+b)^{\nu+1}}.$$

Hence we obtain the formula

$$(36)\quad \int_0^\infty \frac{x^\nu dx}{\{(1+a^2x^2)(1+b^2x^2)\}^{1+\frac{1}{2}\nu}}=\frac{\pi\Gamma(\nu+1)}{2^{\nu+1}\{\Gamma(\tfrac{1}{2}\nu+1)\}^2}\,\frac{1}{(a+b)^{\nu+1}}.$$

This formula may be identified with one given by Cayley and Schlomilch*. For if we write ξ for x^2, $2p$ for a^2+b^2, q for a^2b^2, and $2\rho-1$ for ν, we obtain

$$\int_0^\infty \frac{\xi^{\rho-1}d\xi}{(1+2p\xi+q\xi^2)^{\rho+\frac{1}{2}}}=\frac{\pi\Gamma(2\rho)}{2^{3\rho-1}\{\Gamma(\frac{1}{2}+\rho)^2\}}\,\frac{1}{(p+\sqrt{q})^\rho}$$

$$=2^{-\rho}\sqrt{\pi}\,\frac{\Gamma(\rho)}{\Gamma(\frac{1}{2}+\rho)}\,\frac{1}{(p+\sqrt{q})^\rho}.$$

Now Schlomilch† gave the formula

$$(36)\quad \int_0^\infty \frac{\xi^q d\xi}{(\alpha+\beta\xi+\gamma\xi^2)^{q+\frac{1}{2}}}=\frac{\sqrt{\pi}\,\Gamma(q)}{\Gamma(\frac{1}{2}+q)}\,\frac{1}{(\beta+2\sqrt{\alpha\gamma})^q\sqrt{\gamma}}.$$

If we differentiate with respect to α, write $\rho-1$ for q, and then put $\alpha=1$, $\beta=2p$, $\gamma=q$, we obtain the formula above.

(*b*) If $a=b$, we obtain

$$\int_0^\infty \frac{x^\nu dx}{(1+a^2x^2)^{\nu+\frac{1}{2}(\alpha+\beta)+2}}=\tfrac{1}{2}a^{-\nu-1}\int_0^\infty \frac{\xi^{\frac{1}{2}(\nu-1)}d\xi}{(1+\xi)^{\nu+\frac{1}{2}(\alpha+\beta)+2}}$$

$$=\tfrac{1}{2}a^{-\nu-1}\,\frac{\Gamma\{\frac{1}{2}(\nu+1)\}\,\Gamma\{\frac{1}{2}(\nu+\alpha+\beta+3)\}}{\Gamma\{\nu+\frac{1}{2}(\alpha+\beta)+2\}}$$

Hence

$$(37)\quad \int_0^\infty \psi_\alpha(ax)\,\psi_\beta(ax)\,x^\nu dx$$

$$\frac{\Gamma\{\frac{1}{2}(\nu+1)\}\,\Gamma\{\frac{1}{2}(\nu+\alpha+2)\}\,\Gamma\{\frac{1}{2}(\nu+\beta+2)\}\,\Gamma\tfrac{1}{2}(\nu+\alpha+\beta+3)\}}{8a^{\nu+1}\,\Gamma\{\nu+\frac{1}{2}(\alpha+\beta)+2\}}.$$

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* *Compendium*, ii., p. 280. † *l.c.*

The first type of multiple integral.

§ 7. Let us return now to the formula (16) of § 3. It is equally easy to evaluate an n-ple integral of which this formula is the simplest case. Multiplying the equation

$$\psi_\alpha \sqrt{(c^2+x_1^2+\ldots+x_n^2)} = \int_0^\infty e^{-t^2-(c^2+\Sigma x^2)/4t^2} t^\alpha dt$$

by $$x_1^{\nu_1+1} x_2^{\nu_2+1} \ldots x_n^{\nu_n+1} J_{\nu_1}(2\mu_1 x) \ldots J_{\nu_n}(2\mu_n x),$$

where every $\mu > 0$, and integrating from 0 to ∞ with respect to each variable, we obtain

$$(38)\quad \int_0^\infty \psi_\alpha \{\sqrt{(c^2+\Sigma x^2)}\} \Pi \{x^{\nu+1} J_\nu(2\mu x)\}\, dx$$
$$= 2^{-n} \Pi \mu^\nu \int_0^\infty e^{-t^2(1+\Sigma\mu^2)-(c^2/t^2)} t^{\alpha+2n+2\Sigma\nu} dt$$
$$= 2^{-n} c^{\alpha+2n+1+2\Sigma\nu} \Pi \mu^\nu \psi_{-\alpha-2n-2-2\Sigma\nu} \{c\sqrt{(1+\Sigma\mu^2)}\},$$

if $c > 0$, and

$$(39)\quad \int_0^\infty \psi_\alpha \sqrt{(\Sigma x^2)}\, \Pi \{x^{\nu+1} J_\nu(2\mu x)\}\, dx$$
$$= 2^{-n-1} \Pi \mu^\nu \frac{\Gamma\{\frac{1}{2}(\alpha+1)+n+\Sigma\nu\}}{(1+\Sigma\mu^2)^{\frac{1}{2}(\alpha+1)+n+\Sigma\nu}},$$

if $c = 0$.

The first formula requires that every $\nu > -1$. The second integral is convergent if

$$\int_0 \frac{\Pi x^{2\nu+1} dx}{(\Sigma x^2)^{\frac{1}{2}\alpha}}$$

is convergent—*i.e.* if every $\nu > -1$ and, in addition,

$$\alpha < 2(n+\Sigma\nu).$$

§ 8. Let us, for simplicity, take $n = 2$. Slightly varying our notation, we obtain

$$(40)\quad \int_0^\infty \int_0^\infty \psi_\alpha \{\sqrt{(c^2+x^2+y^2)}\}\, x^{\rho+1} y^{\sigma+1} J_\rho(2\mu x)\, J_\sigma(2\nu x)\, dxdy$$
$$= \tfrac{1}{4} \mu^\rho \nu^\sigma c^{\alpha+2\rho+2\sigma+5} \psi_{-\alpha-2\rho-2\sigma-6} \{c\sqrt{(1+\mu^2+\nu^2)}\},$$

and

$$(41)\quad \int_0^\infty \int_0^\infty \psi_\alpha \{\sqrt{(x^2+y^2)}\}\, x^{\rho+1} y^{\sigma+1} J_\rho(2\mu x)\, J_\sigma(2\nu x)\, dxdy$$
$$= \tfrac{1}{8} \mu^\rho \nu^\sigma \frac{\Gamma\{\frac{1}{2}(\alpha+5)+\rho+\sigma\}}{(1+\mu^2+\nu^2)^{\frac{1}{2}(\alpha+5)+\rho+\sigma}}.$$

In particular, if $\rho = \sigma = -\frac{1}{2}$, we obtain

$$(42)\quad \int_0^\infty \int_0^\infty \psi_\alpha \{\sqrt{(c^2+x^2+y^2)}\} \cos 2\mu x \cos 2\nu y\, dxdy$$
$$= \tfrac{1}{4} \pi c^{\alpha+3} \psi_{-\alpha-4} \{c\sqrt{(1+\mu^2+\nu^2)}\},$$

$$(43)\quad \int_0^\infty\int_0^\infty \psi_\alpha\{\sqrt{(x^2+y^2)}\}\cos 2\mu x\cos 2\nu y\,dxdy = \tfrac{1}{8}\pi\frac{\Gamma\{\frac{1}{2}(\alpha+3)\}}{(1+\mu^2+\nu^2)^{\frac{1}{2}(\alpha+3)}}.$$

If, in addition, $\alpha=0$, we find

$$\begin{aligned}(44)\quad &\int_0^\infty\int_0^\infty e^{-2c\sqrt{(c^2+x^2+y^2)}}\cos 2\mu x\cos\nu y\,dxdy\\ &= \tfrac{1}{2}\sqrt{(\pi)}\,c^3\psi_{-4}\{c\sqrt{(1+\mu^2+\nu^2)}\}\\ &= \tfrac{1}{2}\sqrt{(\pi)}\,\psi_2\{c\sqrt{(1+\mu^2+\nu^2)}\}/(1+\mu^2+\nu^2)^{\frac{3}{2}}\\ &= \frac{\frac{1}{4}\pi}{(1+\mu^2+\nu^2)^{\frac{3}{2}}}e^{-2c\sqrt{(1+\mu^2+\nu^2)}}\{c\sqrt{(1+\mu^2+\nu^2)}+\tfrac{1}{2}\},\end{aligned}$$

and

$$(45)\quad \int_0^\infty\int_0^\infty e^{-2\sqrt{(x^2+y^2)}}\cos 2\mu x\cos 2\nu y\,dxdy = \frac{\frac{1}{8}\pi}{(1+\mu^2+\nu^2)^{\frac{3}{2}}}.$$

These formulæ may be slightly simplified by writing ξ, η for $2x$, $2y$: thus we deduce

$$(46)\quad \int_0^\infty\int_0^\infty e^{-\sqrt{(x^2+y^2)}}\cos\mu x\cos\nu y\,dxdy = \frac{\frac{1}{2}\pi}{(1+\mu^2+\nu^2)^{\frac{3}{2}}}.$$

§ 9. It is interesting to verify these results by a change to polar coordinates. Transforming the integral (40), we obtain

$$\int_0^\infty\int_0^{\frac{1}{2}\pi}\psi_\alpha\sqrt{(c^2+r^2)}\,J_\rho(2\mu r\cos\vartheta)\,J_\sigma(2\nu r\sin\vartheta)\times(\cos\vartheta)^{\rho+1}(\sin\vartheta)^{\sigma+1}r^{\rho+\sigma+3}dr\,d\vartheta.$$

But

$$(47)\quad \int_0^{\frac{1}{2}\pi}J_\rho(2\mu r\cos\vartheta)\,J_\sigma(2\nu r\sin\vartheta)\,(\cos\vartheta)^{\rho+1}(\sin\vartheta)^{\sigma+1}d\vartheta = \frac{\mu^\rho\nu^\sigma}{2r(\mu^2+\nu^2)^{\frac{1}{2}(\rho+\sigma+1)}}J_{\rho+\sigma+1}\{2r\sqrt{(\mu^2+\nu^2)}\},$$

by a formula due to Sonine*. Hence we obtain

$$\tfrac{1}{2}\frac{\mu^\rho\nu^\sigma}{(\mu^2+\nu^2)^{\frac{1}{2}(\rho+\sigma+1)}}\int_0^\infty\psi_\alpha\sqrt{(c^2+r^2)}\,J_{\rho+\sigma+1}\{2r\sqrt{(\mu^2+\nu^2)}\}\,r^{\rho+\sigma+2}dr = \tfrac{1}{4}\mu^\rho\nu^\sigma c^{\alpha+2\rho+2\sigma+5}\psi_{-\alpha-2\rho-2\sigma-6}\{c\sqrt{(1+\mu^2+\nu^2)}\},$$

by (16).

* *Math. Ann.*, bd. 16, p. 36.

§ 10. The formulæ (42) (43) may be generalised as follows: Consider the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\{\sqrt{(\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2)}\}\, e^{(2\mu\xi+2\nu\eta)i}d\xi d\eta,$$

where $\alpha\gamma-\beta^2>0$. Use the substitution

$$\xi = \quad x\cos\Theta + y\sin\Theta,$$
$$\eta = -x\sin\Theta + y\cos\Theta,$$

where $\cos 2\Theta : \sin 2\Theta : 1 :: \gamma-\alpha : 2\beta : \sqrt{\{(\gamma-\alpha)^2+4\beta^2\}}$.

We obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\{\sqrt{(Ax^2+Cy^2)}\}\, e^{(2Mx+2Ny)i}dxdy$$
$$= \frac{4}{\sqrt{(AC)}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\{\sqrt{(X^2+Y^2)}\}\cos\frac{2MX}{\sqrt{A}}\cos\frac{2NY}{\sqrt{C}}dXdY,$$

where
$$A = \tfrac{1}{2}\{\alpha+\gamma-\sqrt{[(\gamma-\alpha)^2+4\beta^2]}\},$$
$$C = \tfrac{1}{2}\{\alpha+\gamma+\sqrt{[(\gamma-\alpha)^2+4\beta^2]}\},$$
$$M = \mu\cos\Theta - \nu\sin\Theta,\quad N = \mu\sin\Theta+\nu\cos\Theta.$$

Hence, in particular,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_s\{\sqrt{(c^2+\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2)}\}\, e^{(2\mu\xi+2\nu\eta)i}d\xi d\eta$$
$$= \frac{4}{\sqrt{(AC)}}\int_0^{\infty}\int_0^{\infty}\psi_s\{\sqrt{(c^2+x^2+y^2)}\}\cos\frac{2Mx}{\sqrt{A}}\cos\frac{2Ny}{\sqrt{C}}\,dxdy,$$
$$= \frac{\pi c^{s+3}}{\sqrt{(\alpha\gamma-\beta^2)}}\psi_{-s-4}\left\{c\sqrt{\left(1+\frac{M^2}{A}+\frac{N^2}{C}\right)}\right\}.$$

But it is easy to verify that

$$\frac{M^2}{A}+\frac{N^2}{C} = \frac{\alpha\nu^2-2\beta\nu\mu+\gamma\mu^2}{\alpha\gamma-\beta^2},$$

and so

$$(48)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_s\{\sqrt{(c^2+\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2)}\}\cos 2\mu\xi\cos 2\nu\eta\, d\xi d\eta$$
$$= \frac{\pi c^{s+3}}{\sqrt{(\alpha\gamma-\beta^2)}}\psi_{-s-4}\left\{c\sqrt{\left(1+\frac{\alpha\nu^2-2\beta\nu\mu+\gamma\mu^2}{\alpha\gamma-\beta^2}\right)}\right\}.$$

Similarly, we deduce

$$(49)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_s\{\sqrt{(\alpha\xi^2+2\beta\xi\eta+\gamma\eta^2)}\}\, e^{(2\mu\xi+2\nu\eta)i}d\xi d\eta$$
$$= \frac{\frac{1}{2}\pi}{\sqrt{(\alpha\gamma-\beta^2)}}\,\frac{\Gamma\{\frac{1}{2}(s+3)\}}{\{1+(\alpha\nu^2-2\beta\nu\mu+\gamma\mu^2)/(\alpha\gamma-\beta^2)\}^{\frac{1}{2}(s+3)}}.$$

In particular, if $s=0$, and we write $\frac{1}{2}x$, $\frac{1}{2}y$ for ξ, η, we obtain

$$(50)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\sqrt{(\alpha x^2+2\beta xy+\gamma y^2)}+(\mu x+\nu y)i}\,dx\,dy = \frac{2\pi}{\sqrt{(\alpha\gamma-\beta^2)}}\left(1+\frac{\alpha\nu^2-2\beta\nu\mu+\gamma\mu^2}{\alpha\gamma-\beta^2}\right)^{-\frac{3}{2}}.$$

§ 11. These formulæ are easily extended to the case of n variables. In fact, the equation

$$(38')\quad \int_0 \psi_s\{\sqrt{(c^2+\Sigma x^2)}\}\,\Pi\cos 2\mu x\,dx = 2^{-n}\pi^{\frac{1}{2}n}c^{s+n+1}\psi_{-s-n-2}\{c\sqrt{(1+\Sigma\mu^2)}\},$$

which we get from (38) by putting every $\nu=-\frac{1}{2}$, at once suggests the formula

$$(51)\quad \int_{-\infty}^{\infty}\psi_s\{\sqrt{(c^2+\Phi)}\}e^{\Sigma 2\mu x i}dx = \frac{\pi^{\frac{1}{2}n}c^{s+n+1}}{\sqrt{\Delta}}\psi_{-s-n-2}\{c\sqrt{(1+\bar{\Phi}_\mu)}\},$$

where Φ denotes the quadratic form

$$a_{11}x_1^2+\ldots+2a_{12}x_1x_2+\ldots,$$

Δ its determinant $|a_{kl}|$, and $\bar{\Phi}_\mu$ the result of writing μ for x in the reciprocal form

$$a'_{11}x_1^2+\ldots+2a'_{12}x_1x_2+\ldots,$$

where

$$a'_{kl}=\frac{1}{\Delta}\frac{\partial\Delta}{\partial a_{kl}}.$$

Similarly (49) and (50) suggest the formulæ

$$(52)\quad \int_{-\infty}^{\infty}\psi_s\{\sqrt{\Phi}\}\,e^{2\Sigma\mu x i}dx = \frac{\frac{1}{2}\pi^{\frac{1}{2}n}}{\sqrt{\Delta}}\,\frac{\Gamma\{\frac{1}{2}(n+s+1)\}}{(1+\bar{\Phi}_\mu)^{\frac{1}{2}(n+s+1)}},$$

$$(53)\quad \int_{-\infty}^{\infty} e^{-\sqrt{\Phi}+\Sigma\mu x i}dx = \frac{2^n\pi^{\frac{1}{2}(n-1)}}{\sqrt{\Delta}}\,\frac{\Gamma\{\frac{1}{2}(n+1)\}}{(1+\bar{\Phi}_\mu)^{\frac{1}{2}(n+1)}}.$$

§ 12. Let us now give a proof of (51). Let

$$x_1=l_{11}\xi_1+l_{12}\xi_2+\ldots,$$
$$x_2=l_{21}\xi_1+\ldots$$

be a real orthogonal substitution, whose determinant is unity, and which reduces $\Sigma a_{12}x_1x_2$ to the form $\Sigma A_1\xi_1^2$, so that A_1, ... are the roots of the equation

$$\begin{vmatrix} a_{11}-\lambda, & a_{12} & , \ldots \\ a_{12} & , a_{22}-\lambda, & \ldots \\ \ldots\ldots & \ldots\ldots & \end{vmatrix}=0.$$

Then the integral on the left-hand side of (51) becomes

$$\int_{-\infty}^{\infty}\psi_s\sqrt{(c^2+\Sigma A^2\xi^2)}\,e^{\Sigma 2M\xi i}d\xi$$

$$=\frac{1}{\sqrt{\Delta}}\int_{-\infty}^{\infty}\psi_s\{\sqrt{(c^2+\Sigma t^2)}\}\,e^{\Sigma(2Mti/\sqrt{A})}dt$$

$$=\frac{\pi^{\frac{1}{2}n}c^{s+n+1}}{\sqrt{\Delta}}\,\psi_{-s-n-2}\left\{c\sqrt{\left(1+\Sigma\frac{M^2}{A}\right)}\right\},$$

where $$M_\nu=\mu_1 l_{1\nu}+\mu_2 l_{2\nu}+\ldots+\mu_n l_{n\nu}.$$

Since
$$\xi_1=l_{11}x_1+l_{21}x_2+\ldots+l_{n1}x_n,$$
$$\xi_2=l_{12}x_1+l_{22}x_2+\ldots+l_{n2}x_n,$$
$$\ldots\ldots\quad\ldots\ldots\ldots\ldots\ldots\ldots$$

the coefficient of $\mu_k\mu_l$ in $\Sigma(M^2/A)$ is the same as that of x_kx_i in $\Sigma(\xi^2/A)$. But $\Sigma(\xi^2/A)=1$, $\Sigma A\xi^2=1$ are reciprocals with respect to $\Sigma\xi^2=1$, and $\Sigma a_{12}x_1x_2=1$, $\Sigma a'_{12}x_1x_2=1$ are reciprocals with respect to $\Sigma x^2=1$: and the transformation which replaces $\Sigma\xi^2$, $\Sigma A\xi^2$ by Σx^2, $\Sigma a_{12}x_1x_2$ will therefore replace $\Sigma(\xi^2/A)$ by $\Sigma a'_{12}x_1x_2$. Hence

$$\Sigma\frac{M^2}{A}\equiv\bar{\Phi}_\mu,$$

and so (51) is established. Similarly for (52) and (53).

The second type of multiple integral.

13. The integral

$$(54)\quad \int_0^\infty\psi_\alpha\sqrt{(c^2+x^2)}\,x^\nu dx=\tfrac{1}{2}\Gamma\{\tfrac{1}{2}(1+\nu)\}\psi_{\alpha+\nu+1}(c)$$

is also the simplest case of an n-ple integral. In fact, if we multiply the equation

$$\psi_\alpha\{\sqrt{(c^2+x_1^2+\ldots+x_n^2)}\}=\int_0^\infty e^{-t^2-(c^2+\Sigma x^2)/t^2}t^\alpha dt$$

by $x_1^{\nu_1}x_2^{\nu_2}\ldots x_n^{\nu_n}$, and integrate from 0 to ∞ with respect to every x, we find, assuming every $\nu>-1$, that

$$(55)\quad \int_0^\infty\psi_\alpha\{\sqrt{(c^2+\Sigma x^2)}\}\,\Pi x^\nu dx=2^{-n}\Pi\Gamma\{\tfrac{1}{2}(1+\nu)\}\psi_{\alpha+n+\Sigma\nu}(c),$$

unless $c=0$, when we obtain

$$(56)\quad \int_0^\infty\psi_\alpha\{\sqrt{(\Sigma x^2)}\}\,\Pi x^\nu dx$$
$$=2^{-n-1}\,\Gamma\{\tfrac{1}{2}(\alpha+n+1+\Sigma\nu)\}\,\Pi\Gamma\{\tfrac{1}{2}(1+\nu)\}.$$

The last formula requires only every $\nu>-1$, if $\alpha>-1$. If $\alpha<-1$ it requires that

$$\int_0\Pi x^\nu(\Sigma x^2)^{\frac{1}{2}(\alpha+1)}dx$$

be convergent, the conditions for which are easily written down (cf. § 7).

In particular let $n=2$. Then

$$(57)\quad \int_0^\infty \psi_\alpha\{\sqrt{(c^2+x^2+y^2)}\}\, x^\mu y^\nu dxdy = \tfrac{1}{4}\Gamma\{\tfrac{1}{2}(1+\mu)\}\,\Gamma\{\tfrac{1}{2}(1+\nu)\}\,\psi_{\alpha+\mu+\nu+2}(c),$$

$$(58)\quad \int_0^\infty \psi_\alpha\{\sqrt{(x^2+y^2)}\}\, x^\mu y^\nu dxdy = \tfrac{1}{8}\Gamma\{\tfrac{1}{2}(1+\mu)\}\,\Gamma\{\tfrac{1}{2}(1+\nu)\}\,\Gamma\{\tfrac{1}{2}(\alpha+\mu+\nu+3)\}.$$

Transforming to polar coordinates and using the equations

$$\int_0^\infty \psi_\alpha \sqrt{(c^2+r^2)}\, r^{\mu+\nu+1} dr = \tfrac{1}{2}\Gamma\{1+\tfrac{1}{2}(\mu+\nu)\}\,\psi_{\alpha+\mu+\nu+2}(c),$$

$$\int_0^{\frac{1}{2}\pi} (\cos\vartheta)^\mu (\sin\vartheta)^\nu d\vartheta = \tfrac{1}{2}\frac{\Gamma\{\frac{1}{2}(1+\mu)\}\,\Gamma\{\frac{1}{2}(1+\nu)\}}{\Gamma\{1+\frac{1}{2}(\mu+\nu)\}},$$

we obtain another equally simple proof.

Some formulæ in multiple series.

§ 14. Cauchy and Fourier proved that, if $f(r)$ is subject to certain restrictions, which have been more precisely analysed by subsequent writers, and

$$\phi(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos rx\, f(r)\, dr,$$

then

$$f(x) = \sqrt{\left(\frac{2}{\pi}\right)} \int_0^\infty \cos rx\, \phi(r)\, dr,$$

and if $\alpha\beta = 2\pi$

$$\sqrt{\alpha}\{\tfrac{1}{2}f(0) + \Sigma_1^\infty f(n\alpha)\} = \sqrt{\beta}\{\tfrac{1}{2}\phi(0) + \Sigma_1^\infty \phi(n\beta)\}.$$

The form of the generalisation of this result to the case of two variables is obvious: it runs,* if

$$(59)\quad \phi(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos rx \cos sy\, f(r,s)\, drds,$$

then

$$(59')\quad f(x,y) = \frac{2}{\pi}\int_0^\infty\int_0^\infty \cos rx \cos sy\, \phi(r,s)\, drds,$$

and if $\alpha\beta = \alpha'\beta' = 2\pi$ then

$$(60)\quad \sqrt{(\alpha\alpha')}\,\Sigma\Sigma' f(m\alpha, n\alpha) = \sqrt{(\beta\beta')}\,\Sigma\Sigma' f(m\beta, n\beta'),$$

* This also I believe to have been stated by Cauchy, but I am unable to give a precise reference.

where

$$\Sigma\Sigma' F(m, n) = \tfrac{1}{4}F(0, 0) + \tfrac{1}{2}\Sigma_1^\infty F(0, n) + \tfrac{1}{2}\Sigma_1^\infty F(m, 0) + \Sigma_1^\infty\Sigma_1^\infty F(m, n).$$

It is evident that the relations between ϕ and f can also be written in the form

$$(59a) \qquad \phi(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(rx+sy)i} f(r, s)\, dr ds,$$

$$(59a') \qquad f(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{(rx+sy)i} \phi(r, s)\, dr ds.$$

Now we have obtained the formula

$$(61) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi_s\{\sqrt{(\alpha\xi^2 + 2\beta\xi\eta + \gamma\eta^2)}\}\, e^{(\mu\xi+\nu\eta)i} d\xi d\eta = \frac{\frac{1}{2}\pi}{\sqrt{(\alpha\gamma - \beta^2)}} \frac{\Gamma\{\frac{1}{2}(s+3)\}}{\left[1 + \frac{1}{4}\{(\alpha\nu^2 - 2\beta\nu\mu + \gamma\mu^2)/(\alpha\gamma - \beta^2)\}\right]^{\frac{1}{2}(s+3)}},$$

so that the functions

$$f(x, y) = \psi_s\{\sqrt{(\alpha x^2 + 2\beta xy + \gamma y^2)}\},$$

$$\phi(x, y) = \frac{1}{4\sqrt{(\alpha\gamma - \beta^2)}} \frac{\Gamma\{\frac{1}{2}(s+3)\}}{\left[1 + \frac{1}{4}\{(\alpha y^2 - 2\beta yx + \gamma x^2)/(\alpha\gamma - \beta^2)\}\right]^{\frac{1}{2}(s+3)}}$$

are related in the manner described above. We are thus led to anticipate the equation

$$(62) \quad \psi_s\{\sqrt{(\alpha\xi^2 + 2\beta\xi\eta + \gamma\eta^2)}\} = \frac{\Gamma\{\frac{1}{2}(s+3)\}}{8\pi\sqrt{(\alpha\gamma - \beta^2)}} \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(\mu\xi+\nu\eta)i} d\mu d\nu}{\left[1 + \frac{1}{4}\{(\alpha\nu^2 - 2\beta\nu\mu + \gamma\mu^2)/(\alpha\gamma - \beta^2)\}\right]^{\frac{1}{2}(s+3)}},$$

or, writing x, y for ξ, η and putting 2μ, 2ν again for μ, ν,

$$(63) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(2\mu x+2\nu y)i} d\mu d\nu}{(\alpha\gamma - \beta^2 + \alpha\nu^2 - 2\beta\nu\mu + \gamma\mu^2)^{\frac{1}{2}(s+3)}} = \frac{2\pi}{\Gamma\{\frac{1}{2}(s+3)\}\,(\alpha\gamma - \beta^2)^{\frac{1}{2}s+1}} \psi_s\{\sqrt{(\alpha x^2 + 2\beta xy + \gamma y^2)}\},$$

and, in particular, if $s = 0$,

$$(64) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{(2\mu x+2\nu y)i} d\mu d\nu}{(\alpha\gamma - \beta^2 + \alpha\nu^2 - 2\beta\nu\mu + \gamma\mu^2)^{\frac{3}{2}}} = \frac{2\pi}{\alpha\gamma - \beta^2} e^{-2\sqrt{(\alpha x^2 + 2\beta xy + \gamma y^2)}}.$$

If $\alpha=\gamma=1$, $\beta=0$, we find

$$(65)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(2\mu x+2\nu y)i}}{(1+\mu^2+\nu^2)^{\frac{1}{2}(s+3)}}\,d\mu d\nu = \frac{2\pi}{\Gamma\{\frac{1}{2}(s+3)\}}\,\psi_s\{\sqrt{(x^2+y^2)}\},$$

$$(66)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(2\mu x+2\nu y)i}}{(1+\mu^2+\nu^2)^{\frac{3}{2}}}\,d\mu d\nu = 2\pi e^{-2\sqrt{(x^2+y^2)}}.$$

The last two formulæ may be verified by transformation to polar coordinates. Putting

$$\mu=\rho\cos\phi,\quad \nu=\rho\sin\phi,$$

we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{(2\mu x+2\nu y)i}}{(1+\mu^2+\nu^2)^{\frac{1}{2}(s+3)}}\,d\mu d\nu$$

$$=4\int_0^{\infty}\int_0^{\infty}\frac{\cos 2\mu x\cos 2\nu y\,d\mu d\nu}{(1+\mu^2+\nu^2)^{\frac{1}{2}(s+3)}}$$

$$=4\int_0^{\infty}\frac{\rho d\rho}{(1+\rho^2)^{\frac{1}{2}(s+3)}}\int_0^{\frac{1}{2}\pi}\cos(2x\rho\cos\phi)\cos(2y\rho\sin\phi)\,d\phi,$$

$$=2\pi\int_0^{\infty}J_0\{2\rho\sqrt{(x^2+y^2)}\}\,\frac{\rho d\rho}{(1+\rho^2)^{\frac{1}{2}(s+3)}},$$

by Sonine's formula (47), or

$$=\frac{2\pi}{\Gamma\{\frac{1}{2}(s+3)\}}\,\psi_s\{\sqrt{(x^2+y^2)}\},$$

by another formula also due to Sonine*.

§ 15. Cauchy's theorem also leads us to anticipate the equation

$$\kappa\Sigma\Sigma'\psi_s\{\kappa\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\}$$

$$=\frac{\kappa'\Gamma\{\frac{1}{2}(s+3)\}}{4\sqrt{(\alpha\gamma-\beta^2)}}\,\Sigma\Sigma'\frac{1}{[1+\frac{1}{4}\kappa'^2\{(\alpha n^2-2\beta mn+\gamma m^2)/(\alpha\gamma-\beta^2)\}]^{\frac{1}{2}(s+3)}},$$

where $\kappa\kappa'=2\pi$, and a similar equation obtained by changing the sign of β. From these we deduce

$$(67)\quad \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\psi_s\{\kappa\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\}=\frac{\pi\Gamma\{\frac{1}{2}(s+3)\}}{2\kappa^2\sqrt{(\alpha\gamma-\beta^2)}}$$

$$\times\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\frac{1}{[1+\{(\pi^2/\kappa^2)(\alpha n^2-2\beta mn+\gamma m^2)/(\alpha\gamma-\beta^2)\}]^{\frac{1}{2}(s+3)}}.$$

* *l.c.*, p. 50.

The series on the right is convergent if $s>-1$, and all the ψ-functions on the left have a meaning under the same condition. We therefore suppose $s>-1$, and we can show that the equation is then, in fact, true as follows.

We have

$$\psi_s\{\kappa\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\}=\int_0^\infty e^{-t^2-(\alpha m^2+2\beta mn+\gamma n^2)\kappa^2/t^2}t^s dt,$$

which holds for all pairs of integral values of m and n, including $m=0$, $n=0$, provided $s>-1$; and it is easy to show that, if this condition is satisfied,

$$(68)\quad \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\psi_s\{\kappa\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\} = \int_0^\infty e^{-t^2}t^s dt\,\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}e^{-(\alpha m^2+2\beta mn+\gamma n^2)\kappa^2/t^2}.$$

This last series may, in the notation of Krazer and Prym, be written*

$$\vartheta\begin{vmatrix}0,&0\\0,&0\end{vmatrix}(0,\ z)_\phi,$$

where $z=\kappa^2/\pi t^2$ and ϕ denotes the quadratic form

$$\alpha m^2+2\beta mn+\gamma n^2.$$

But

$$(69)\quad \vartheta\begin{vmatrix}0,&0\\0,&0\end{vmatrix}(0,\ z)_\phi=\frac{1}{z\sqrt{(\alpha\gamma-\beta^2)}}\vartheta\begin{vmatrix}0,&0\\0,&0\end{vmatrix}(0,\ 1/z)_\Phi$$

where Φ denotes the reciprocal form

$$(\gamma m^2-2\beta mn+\alpha n^2)/(\alpha\gamma-\beta^2).$$

Hence

$$(70)\quad \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}e^{-(\alpha m^2+2\beta mn+\gamma n^2)\kappa^2/t^2} = \frac{\pi t^2}{\kappa^2\sqrt{(\alpha\gamma-\beta^2)}}\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}e^{-(\gamma m^2-2\beta mn+\alpha n^2)\pi^2t^2/(\alpha\gamma-\beta^2)\kappa^2}.$$

Hence we obtain the integral

$$\frac{\pi}{\kappa^2\sqrt{(\alpha\gamma-\beta^2)}}\int_0^\infty e^{-t^2}t^{s+2}dt\,\Sigma\Sigma e^{-(\gamma m^2-2\beta mn+\alpha n^2)\pi^2t^2/(\alpha\gamma-\beta^2)\kappa^2}.$$

Finally, since

$$\int_0^\infty e^{-t^2\{1+(\gamma m^2-2\beta mn+\alpha n^2)\pi^2/(\alpha\gamma-\beta^2)\kappa^2\}}t^{s+2}dt = \tfrac{1}{2}\Gamma\{\tfrac{1}{2}(s+3)\}\left\{1+\frac{(\gamma m^2-2\beta mn+\alpha n^2)\pi^2}{(\alpha\gamma-\beta^2)\kappa^2}\right\}^{-\frac{1}{2}(s+3)},$$

* Epstein, *Math. Annalen*, bd. 56, p. 623.

we arrive at the equation

$$(71)\quad \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\psi_s\{\kappa\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\}$$
$$=\frac{\frac{1}{2}\pi\Gamma\{\frac{1}{2}(s+3)\}}{\kappa^2\sqrt{(\alpha\gamma-\beta^2)}}\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\left\{1+\frac{(\gamma m^2-2\beta mn+\alpha n^2)\pi^2}{(\alpha\gamma-\beta^2)\kappa^2}\right\}^{-\frac{1}{2}(s+3)},$$

which is the equation (67) above. The series is convergent if $s>-1$: and the inversion of the order of summation and integration is easily justified under this condition.

In particular, for $s=0$, we obtain

$$(72)\quad \Sigma\Sigma e^{-2\kappa\sqrt{(\alpha m^2+2\beta mn+\alpha n^2)}}$$
$$=\frac{\pi}{2\kappa^2\sqrt{(\alpha\gamma-\beta^2)}}\Sigma\Sigma\left\{1+\frac{(\gamma m^2-2\beta mn+\alpha n^2)\pi^2}{(\alpha\gamma-\beta^2)\kappa^2}\right\}^{-\frac{5}{2}}.$$

If $\alpha=\gamma=1$, $\beta=0$, $\kappa=a\pi$, we find

$$(73)\quad \Sigma\Sigma\psi_s\{a\pi\sqrt{(m^2+n^2)}\}=\frac{a^{s+1}\Gamma\{\frac{1}{2}(s+3)\}}{2\pi}\Sigma\Sigma\frac{1}{(a^2+m^2+n^2)^{\frac{1}{2}(s+3)}},$$

$$(74)\quad \Sigma\Sigma e^{-2a\pi\sqrt{(m^2+n^2)}}=\frac{a}{2\pi}\Sigma\Sigma\frac{1}{(a^2+m^2+n^2)^{\frac{3}{2}}}.$$

§ 16. The equations of the preceding paragraph may be connected, in a way that is not without interest, with the general formulæ for the transformation of multiple zeta-functions recently given by Epstein*.

Let

$$(75)\quad \chi_s(c)=\Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\psi_s\{c\pi\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\},$$

and consider the integral

$$\int_0^{\infty}\chi_s(c)\,c^{\nu-1}dc.$$

Under certain conditions we shall have

$$\int_0^{\infty}\chi_s(c)\,c^{\nu-1}dc=\Sigma\Sigma\int_0^{\infty}\psi_s\{c\pi\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\}\,c^{\nu-1}dc.$$

But if $s>-1$, $\nu>0$, or if $s<-1$, $s+\nu+1>0$, we have, by (29),

$$\int_0^{\infty}\psi_s\{c\pi\sqrt{(\alpha m^2+2\beta mn+\gamma n^2)}\}\,c^{\nu-1}dc$$
$$=\tfrac{1}{4}\Gamma(\tfrac{1}{2}\nu)\,\Gamma\{\tfrac{1}{2}(s+\nu+1)\}/\pi^{\nu}(\alpha m^2+2\beta mn+\gamma n^2)^{\frac{1}{2}\nu}.$$

Also the series

$$Z(\nu)=\Sigma\Sigma\frac{1}{(\alpha m^2+2\beta mn+\gamma n^2)^{\frac{1}{2}\nu}}$$

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* *Math. Annalen*, l.c.

is convergent if $\nu > 2$. Hence we have, under certain conditions,

$$(76)\quad \int_0^\infty \chi_s(c)\, c^{\nu-1} dc = \tfrac{1}{4}\Gamma(\tfrac{1}{2}\nu)\,\Gamma\{\tfrac{1}{2}(s+\nu+1)\}\,\pi^{-\nu} Z(\nu).$$

But, by (67), we have also, if $s+1>0$,

$$\chi_s(c) = \frac{\frac{1}{2}\Gamma\{\frac{1}{2}(s+3)\}}{\pi c^2\sqrt{(\alpha\gamma-\beta^2)}} \Sigma_{-\infty}^{\infty}\Sigma_{-\infty}^{\infty}\left\{1+\frac{\gamma m^2 - 2\beta mn + \alpha n^2}{(\alpha\gamma-\beta^2)\,c^2}\right\}^{-\frac{1}{2}(s+3)}.$$

The integral

$$\int_0^\infty \left(1+\frac{k^2}{c^2}\right)^{-\frac{1}{2}(s+3)} c^{\nu-3} dc$$

is convergent if $\nu < 2$, $s+\nu+1>0$: under these conditions it is equal to

$$k^{\nu-2}\int_0^\infty \frac{\xi^{\nu+3}\,d\xi}{(1+\xi^2)^{\frac{1}{2}(s+3)}} = \tfrac{1}{2}k^{\nu-2}\int_0^\infty \frac{t^{\frac{1}{2}(\nu+s-1)}}{(1+t)^{\frac{1}{2}(s+3)}}\,dt$$

$$= \tfrac{1}{2}k^{\nu-2}\frac{\Gamma(1-\frac{1}{2}\nu)\,\Gamma\{\frac{1}{2}(\nu+s+1)\}}{\Gamma\{\frac{1}{2}(s+3)\}},$$

and so

$$(77)\quad \int_0^\infty \left\{1+\frac{\gamma m^2-2\beta mn+\alpha n^2}{(\alpha\gamma-\beta^2)\,c^2}\right\}^{-\frac{1}{2}(s+3)} c^{\nu-3}dc$$

$$= \tfrac{1}{2}\frac{\Gamma(1-\frac{1}{2}\nu)\,\Gamma\{\frac{1}{2}(\nu+s+1)\}}{\Gamma\{\frac{1}{2}(s+3)\}}\left(\frac{\alpha\gamma-\beta^2}{\gamma m^2-2\beta mn+\alpha n^2}\right)^{1-\frac{1}{2}\nu}.$$

The series

$$\overline{Z}(2-\nu) = \Sigma\Sigma\frac{1}{(\gamma m^2-2\beta mn+\alpha n^2)^{1-\frac{1}{2}\nu}}$$

is convergent if $\nu<0$. And it is easy to show that if $\nu<0$, $\nu+s+1>0$ (conditions which involve $s+1>0$),

$$(78)\quad \int_0^\infty \chi_s(c)\, c^{\nu-1}dc$$

$$= \frac{\frac{1}{2}\Gamma\{\frac{1}{2}(s+3)\}}{\pi\sqrt{(\alpha\gamma-\beta^2)}}\Sigma\Sigma\int_0^\infty\left\{1+\frac{\gamma m^2-2\beta mn+\alpha n^2}{(\alpha\gamma-\beta^2)\,c^2}\right\}^{-\frac{1}{2}(s+3)} c^{\nu-3}dc$$

$$= \frac{1}{4\pi}\Gamma(1-\tfrac{1}{2}\nu)\,\Gamma\{\tfrac{1}{2}(\nu+s+1)\}(\alpha\gamma-\beta^2)^{\frac{1}{2}(1-\nu)}\overline{Z}(2-\nu).$$

Thus the integral

$$\int_0^\infty \chi_s(c)\, c^{\nu-1}dc$$

is, for some values of ν, equal to

$$\tfrac{1}{4}\Gamma(\tfrac{1}{2}\nu)\,\Gamma\{\tfrac{1}{2}(\nu+s+1)\}\,\pi^{-\nu}Z(\nu),$$

and for others equal to

$$\frac{1}{4\pi}\Gamma(1-\tfrac{1}{2}\nu)\,\Gamma\{\tfrac{1}{2}(\nu+s+1)\}\,(\alpha\gamma-\beta^2)^{\frac{1}{2}(1-\nu)}\overline{Z}(2-\nu).$$

This result suggests the truth of the functional relation

$$(79)\quad \Gamma(\tfrac{1}{2}\nu)\,\pi^{1-\nu}Z(\nu)=(\alpha\beta-\gamma^2)^{\frac{1}{2}(1-\nu)}\Gamma(1-\tfrac{1}{2}\nu)\,\overline{Z}(2-\nu),$$

which is, as a matter of fact, included in Epstein's general results. To show that the two formulæ which represent the integral in different parts of the range of variation of ν do represent the same analytic function would, however, lead us too far—nor would this be by any means the best way of establishing the truth of the functional relation, for which we must refer to Epstein's memoir.

ON INFINITE SUMS AND PRODUCTS OF CARDINAL NUMBERS.

By Philip E. B. Jourdain.

THIS paper is occupied with a proof that a relation of inequality holds between certain infinite products and certain connected infinite sums of cardinal numbers. The theorems of Cantor* and König† appear as special cases of my theorems, which were obtained by attempting to give the most general extension possible to the method of proving inequalities introduced by Cantor. This method may be thus described. If we have two aggregates M and N such that N is known to be similar to a part of M, and hence, by the Schröder-Bernstein theorem‡, that

$$\mathfrak{m}\geqq\mathfrak{n},$$

where $\mathfrak{m}$ and $\mathfrak{n}$ are the cardinal numbers of M and N respectively, then Cantor's method of proving that

$$\mathfrak{m}>\mathfrak{n},$$

or that the equality is excluded, consists in the definition of an element of M that is not included in any (1, 1) correspondence

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* "Ueber eine elementare Frage der Mannigfaltigkeitslehre," *Jahresber. der Deutsch. Math.-Ver.*, Bd. i., 1892, pp. 75–78.

† "Zum Kontinuum-Problem," *Verh. des* iii. *Internat. Math. Kongresses* in Heidelberg in 1904 [1905], pp. 144–147.

‡ Cf. *Quart. Jour. of Math.*, vol. xxxviii., 1907, pp. 355–356.

ϕ of N with a part of M. The definition of this element is, of course, given in terms of ϕ^*.

Now, although Cantor's argument did not require the multiplicative axiom,†—and perhaps his choice of the theorem

$$2^{\mathfrak{a}} > \mathfrak{a}$$

instead of (say) $\mathfrak{m}^{\mathfrak{a}} > \mathfrak{a}$‡,

as well as his preference§ for an independent definition of the operation of raising to a transfinite power to the method of connecting it with the conception of a transfinite product‖ were dictated by a wish to avoid¶ this axiom,—the conception of a transfinite product does, and consequently König's theorem, which is based on a method of conclusion like Cantor's, that

$$\prod_{\nu<\omega} \mathfrak{m}_\nu > \sum_{\nu<\omega} \mathfrak{m}_\nu,$$

where $\mathfrak{m}_\nu < \mathfrak{m}_{\nu+1}$,

and his deduction that

$$\aleph_{\gamma+\omega}^{\aleph_0} > \aleph_{\gamma+\omega},$$

where γ is any ordinal number, do also. This axiom, a case of which is necessary to make the conception of a transfinite product legitimate, and which, as Zermelo** has pointed out,

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* Cf. Russell, "The Principles of Mathematics," vol. I., Cambridge, 1902, pp. 366–368, and *Quart. Jour. of Math.*, vol. XXXVIII, 1907, p. 359, on the difficulty to which this argument gives rise in some cases.

† For details as to this axiom, see *Quart. Jour. of Math.*, vol. XXXVIII., 1907, pp. 360–366.

‡ Schönflies ("Die Entwickelung der Lehre von der Punkt mannigfaltigkeiten, Leipzig, 1900, p. 26) proved the theorem $\mathfrak{m}^{\mathfrak{m}} > \mathfrak{m}$ ($\mathfrak{m}$ being transfinite) in a way similar to Cantor's proof $2^{\mathfrak{m}} > \mathfrak{m}$, but by using, what is necessary, the above axiom. He silently assumed this axiom.

* *Math. Ann.*, Bd. XLVI., 1895, p. 487.

† The connection of Cantor's definition of exponentiation with that of multiplication, which was not pointed out by Cantor, perhaps for the above reason, was remarked by Schönflies (*op. cit.*, p. 9) and Whitehead (*Amer. Jour. of Math.*, vol. XXIV., 1902, pp. 383–385; see *Quart. Jour. of Math.*, vol XXXVIII., 1907, p. 364), without, however, noticing the need of the axiom.

‡ Cantor did not succeed in avoiding it in his proof that every transfinite aggregate contains $\aleph_0$ terms at least (see *Quart. Jour. of. Math.*, vol. XXXVIII., 1907, pp. 360–361).

** *Math. Ann.*, Bd. LIV., 1904, pp. 514–516. An equivalent of Zermelo's axiom results from the (less general) axiom, necessary for infinite products to be non-zero, that a "multiplicative class" of the members of a class of *mutually exclusive* classes exists, by simply missing out the words '*mutually exclusive*' (see *Quart. Jour. of Math.*, vol. XXXVIII., 1907, p. 364). Thus Zermelo's remark that his axiom is equivalent to the former (less general) axiom is not correct, or, at least, is as yet unjustified.

is equivalent to the assumption that any aggregate can be well-ordered, is here assumed explicitly (§ I.).

After developing at some length (§ II.) the proof that

$$\prod_{\gamma} \mathfrak{m}_\gamma \geqq \sum_{\gamma} \mathfrak{m}_\gamma,$$

I prove (§§ III., IV.)

$$\prod_{\gamma} \mathfrak{m}_\gamma > \sum_{\gamma} \mathfrak{m}_\gamma,$$

under suppositions very much more general than those of König's theorem; and, from this,

$$(\mathfrak{n}+1)^{\mathfrak{a}} > \mathfrak{a}.\mathfrak{n},$$

where $\mathfrak{n}$ is finite, and hence

$$\mathfrak{b}^{\mathfrak{a}} > \mathfrak{a},$$

where $\mathfrak{b}$ is any cardinal number (the generalised Cantor's theorem), and also a generalisation of König's most important result follow. This last generalisation is: If ζ is any Limes-number and ξ is an ordinal such that $\aleph_\xi$ is the cardinal number of an aggregate which can be well-ordered in a series of type ζ, then

$$\aleph_{\kappa+\zeta}^{\aleph_\xi} > \aleph_{\kappa+\zeta},$$

where κ is any ordinal number.

I.

If u is a class (or aggregate) of mutually exclusive classes M_γ, no M being null, while the cardinal number of M_γ is denoted by $\mathfrak{m}_\gamma$, the *sum* (s) of all these M_γ's, or

$$\sum_{\gamma} \mathfrak{m}_\gamma \quad (M_\gamma \epsilon u),$$

is defined* to be the cardinal number of the logical sum† of u. In particular, if these $\mathfrak{m}_\gamma$'s can be arranged in a well-ordered series of type α, we denote s by

$$\sum_{\gamma<\alpha} \mathfrak{m}_\gamma.$$

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* This is a simple generalisation of Cantor's definition in *Math. Ann.*, Bd. XLVI., 1895, p. 485.

† Cf. *Quart. Jour. of Math.*, vol. XXXVIII., 1907, p. 357. The logical sum of u is the class of x's such that the complex of the propositions: x is a member of z, and: z is a member of u is true. If u has *two* members M_1 and M_2, the logical sum of u is what Cantor called the "Vereinigungsmenge" of M_1 and M_2 (cf. also Schönflies, *op. cit.*, p. 6).

Arising from the conception of a sum we get* that of a *product* of these $\mathfrak{m}_\gamma$'s. We select any one element m_γ out of each M_γ and form the complex μ of all these m_γ's. By selecting another element, say $m_\gamma^{(2)}$, from M_γ instead of m_γ, we form a different μ; and we proceed so for all the other terms of the original μ, so that each μ contains one and only one, term of each M_γ. The cardinal number $\mathfrak{p}$ of the whole class of these μ's is defined as the *product* of the $\mathfrak{m}_\gamma$'s and denoted

$$\prod_\gamma \mathfrak{m}_\gamma \quad (M_\gamma \varepsilon u).$$

In particular, if the m_γ's can form, as above, a well-ordered series of type α, then each μ has the form

$$\{m_1, m_2, \ldots, m_\gamma, \ldots\}, \quad (\gamma < \alpha),$$

where each m_γ is some one of the terms of M_γ, and $\mathfrak{p}$ is written

$$\prod_{\gamma<\alpha} \mathfrak{m}_\gamma.$$

If all the $\mathfrak{m}_\gamma$'s are equal to $\mathfrak{m}$, this $\mathfrak{p}$ becomes 'exponentiated by' $\mathfrak{a}$, or

$$\mathfrak{m}^{\mathfrak{a}},$$

where $\mathfrak{a}$ is the cardinal number of u, or of any class which can be arranged in type α, as the case may be.

The possibility, in general, of the selection referred to is here accepted as an axiom. This axiom enters, in some one of its forms, whenever we need to make a selection of an infinity of terms from some class, when no law or rule which we can exactly describe in a finite number of terms exists which governs this selection. Thus, an arbitrary choice of an infinity of members (say, in succession,

$$m_1, m_2, \ldots, m_\omega, m_{\omega+1}, \ldots\dagger),$$

for the purpose of proving that any aggregate can be well-ordered, requires this axiom, as also do certain statements of the Heine-Borel theorem‡, and the theory of the product of an infinity of ordinal types§.

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* Cantor, *loc. cit.*, p. 485; Schönflies, *op. cit*, pp. 7, 9; Whitehead, *loc. cit.*, pp. 384–385; and König, *loc. cit.*, p. 144. Each μ is an element of what Cantor called (when there are *two* M_γ's) the "Verbindungsmenge" of the M_γ's.

† As in my paper in *Math. Ann.*, Bd. LX., 1905, p. 468. I did not then fully realise that a transfinite process of selection required an axiom which could be expressed otherwise than by the assumption that we could conceive a transfinite process, more or less analogous to our usual finite one.

‡ See my paper in the *Mess. of Math.*, vol. XXXVI., August, 1906, pp. 64–65.

§ See my paper "The Multiplication of an Infinity of Ordinal Types," *Mess. of Math.*, vol. XXXVI., May, 1906, pp. 13–16.

II.

To prove that

$$\mathfrak{m}^{\mathfrak{a}} \geqq \mathfrak{a},—$$

a necessary preliminay to proving that

$$\mathfrak{m}^{\mathfrak{a}} > \mathfrak{a}—,$$

that is, to prove that there is a part of the $(A/M)^*$ considered which is similar to the A considered, we take one of the elements (μ) of the (A/M) and manufacture, by putting for each m in turn† of μ another m' (also another element of M), while keeping all the other elements of μ the same, an aggregate of elements of (A/M) similar to A.

On the other hand, to prove that

$$\prod_{\gamma} \mathfrak{m}_\gamma \geqq \mathfrak{m} \sum_{\gamma} \mathfrak{n}_\gamma,$$

we take an element μ of P, P being the aggregate of cardinal number $\prod_{\gamma} \mathfrak{m}_\gamma$ which we consider, each term of μ being one of the elements of one of the M's. Select a particular term of μ, and, keeping all the other terms of μ constant, vary this element through the whole of the M to which it belongs; and do this, in turn, for every term of μ. By this means, if, by not counting the μ with which we start, we avoid getting the same term twice over, we get

$$\sum_{\gamma} (\mathfrak{m}_\gamma - 1)$$

elements of P; which evidently becomes $\sum_{\gamma} \mathfrak{m}_\gamma$ if every $\mathfrak{m}_\gamma$ is transfinite‡.

It is advantageous to modify slightly this process as follows. To each M, adjoin an element m_0 and form the

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* Cantor's notation (*loc. cit.*, p. 487; Schönflies, *op. cit.*, p. 8) for the "Belegungsmenge" of A with M, whose cardinal number is $\mathfrak{m}^{\mathfrak{a}}$.

† These words are not meant to imply that μ can be *well-ordered*; the meaning is quite obvious.

‡ König's theorem (stated, not proved, in a note on *loc. cit.*, pp. 144–145) that

$$\prod_{\gamma<\omega} m_\gamma \geqq \sum_{\gamma<\omega} m_\gamma$$

if, and only if, $\prod_{\gamma<\omega} m_\gamma$ is transfinite, results from the relation

$$\prod_{\gamma<\omega} m_\gamma \geqq \sum_{\gamma<\omega} (m_\gamma - 1),$$

which is equivalent to the relation

$$\mathfrak{p} + \aleph_0 \geqq \mathfrak{s};$$

and $\mathfrak{p} + \aleph_0 = \mathfrak{p}$ if, and only if, $\mathfrak{p}$ is transfinite.

The number $m_\gamma - 1$ is *always* unique, though subtraction is not, in general, unique when the numbers are transfinite.

complex μ_0; for each m_θ in μ_0, in turn, substitute the various elements in turn of the M to which that particular m_θ is adjoined. In this way we get an aggregate similar to S contained in a P', which is like P except that there is an extra element μ_θ, each μ has one term more, and each M has one term (m_θ) more. Thus $\mathfrak{p}'$, the cardinal number of P' is

$$\prod_\gamma (\mathfrak{m}_\gamma + 1),$$

and we have

$$\mathfrak{p}' \geqq \mathfrak{s}. \tag{1}$$

If every $\mathfrak{m}_\gamma$ is 1,* and there are $\mathfrak{a}$ M's, (1) gives

$$2^{\mathfrak{a}} \geqq \mathfrak{a},$$

the first stage of Cantor's theorem; while, if every M is transfinite so that

$$\mathfrak{m}_\gamma + 1 = \mathfrak{m}_\gamma,$$

we have

$$\mathfrak{p}' = \mathfrak{p}, \text{ and hence } \mathfrak{p} \geqq \mathfrak{s},$$

we have the first stage of a generalised König's theorem—König's theorem applying to the case of the product and sum being

$$\prod_{\gamma < \omega} \mathfrak{m}_\gamma \text{ and } \sum_{\gamma < \omega} \mathfrak{m}_\gamma$$

respectively.

In this case, then, we obtain the relation between $\mathfrak{p}$ and $\mathfrak{s}$ from the relation $\mathfrak{p}' \geqq \mathfrak{s}$, which latter *always* holds. But the most general relation between $\mathfrak{p}$ and $\mathfrak{s}$ is obtained by an easy generalisation of that result of König's referred to in the preceding note†. This generalisation is

$$\mathfrak{p} \geqq \mathfrak{s}$$

if, and only if, $\mathfrak{p}$ is such that

$$\mathfrak{p} + \mathfrak{a} = \mathfrak{p}, \tag{2}$$

there being $\mathfrak{a}$ factors in $\mathfrak{p}$ (and summands in $\mathfrak{s}$).

If we admit the multiplicative axiom, which implies that any transfinite cardinal number is an Aleph, we know‡ that (2) holds when, and only when $\mathfrak{p} \geqq \mathfrak{a}$.

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* Note that, in this case, $\mathfrak{p}$ is finite, and $\mathfrak{s}$ is not.

† We also have $\mathfrak{s}^{\mathfrak{a}} \geqq \mathfrak{p}$, in generalisation of König's $\mathfrak{s}\aleph_0 \geqq \mathfrak{p}$.

‡ See *Phil. Mag.*, Jan., 1904, pp. 72–74.

III.

We will now suppose a definite (1, 1) correspondence ϕ established between S and a part of P', and will enquire under what conditions an element belonging to P', but corresponding to no element of S, can be defined.

For this purpose, consider an element $m_\gamma^{(1)}$ (also an element of M_γ) of S, and let the element $\mu^{(1)}$ of P' correspond to it, so that

$$\mu^{(1)} = \phi\,(m_\gamma^{(1)}).$$

In $\mu^{(1)}$ seek out the term which is an element of M_γ, or the adjoined element $m_\gamma^{(0)}$. This term is not, in general, either $m_\gamma^{(1)}$ or $m_\gamma^{(0)}$ unless each M_γ has only one element, when it must be one of the two mentioned. But if this is the case, by choosing, instead of $m_\gamma^{(1)}$, the adjoined element $m_\gamma^{(0)}$ of M_γ, and choosing $m_\gamma^{(0)}$ instead of $m_\gamma^{(1)}$, and doing this for all the M_γ's, we get a $\mu^{(0)}$ of P' corresponding to no element of S; and hence, since every $\mathfrak{m}_\gamma$ reduces to 1, we get Cantor's theorem

$$2^{\mathfrak{a}} > \mathfrak{a}.$$

But further, if all the $\mathfrak{m}_\gamma$'s are finite so that

$$\mathfrak{m}_\gamma + 1 > \mathfrak{m}_\gamma,$$

and we admit the multiplicative axiom, we can find, by this axiom, an element of P' corresponding to no element of S, and hence can assert

$$\mathfrak{p}' > \mathfrak{s}. \tag{3}$$

If every $\mathfrak{m}_\gamma = \mathfrak{n}$ (a finite cardinal number) (3) becomes

$$(\mathfrak{n}+1)^{\mathfrak{a}} > \mathfrak{a}\,.\,\mathfrak{n}. \tag{4}$$

From this inequality, we can deduce the most general form* of Cantor's theorem. Let $\mathfrak{b}$ be any cardinal number greater than 1, we have, putting $\mathfrak{n} = 1$,

$$\mathfrak{b}^{\mathfrak{a}} \geqq 2^{\mathfrak{a}} > \mathfrak{a}.$$

IV.

Consider, now, the case of every M_γ being transfinite, so that

$$\mathfrak{m}_\gamma + 1 = \mathfrak{m}_\gamma,$$

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* Of course, if we admit the multiplicative axiom, $\mathfrak{a}\,.\,\mathfrak{n} = \mathfrak{a}$, and (4) results from Cantor's theorem; but the more general *method* here adopted may be not without interest.

and the above method no longer applies, owing to the fact that, in the ϕ between S and a part of P', with which we start all the ϕ-correlates of the elements of M in S, that is to say, all the μ's in which each term $m_\gamma^{(3)}$ is chosen, in all possible ways, from the aggregate $\{M_\gamma, m_\gamma^{(0)}\}$, form a class in which we cannot say, in general, that there is an element of $\{M_\gamma, m_\gamma^{(0)}\}$ not used in this correlation. In this case, if there is another M_ξ such that

$$\mathfrak{m}_\xi > \mathfrak{m}_\gamma,$$

we can conclude that not all the elements of $\{M_\xi, m_\xi^{(0)}\}$ are used up, in the place marked by m_ξ, in the ϕ-correlates of M_γ.

If, then, every M_γ in S has such an aggregate M_ξ*, of greater cardinal number, belonging to it, we can select, by means of the multiplicative axiom, an element out of each M_ξ, and construct a μ out of these elements, so that this μ, although a member of P', corresponds to no member of S in the ϕ considered. Hence, if all the M_γ's are transfinite, then *if, and only if*, for each M_γ there is, by a (1, 1) correspondence of all the M's with a part of themselves, a M_ξ such that

$$\mathfrak{m}_\xi > \mathfrak{m}_\gamma,$$

can we assert that

$$\prod_\gamma \mathfrak{m}_\gamma > \sum_\gamma \mathfrak{m}_\gamma.$$

In particular, these conditions are fulfilled if the $\mathfrak{m}$'s are Alephs of which there is no greatest. Then they may be arranged in a well-ordered series, with no last term, of cardinal numbers which increase in magnitude, for example

$$\aleph_\kappa, \aleph_{\kappa+1}, \aleph_{\kappa+2}, \cdots, \aleph_{\kappa+\gamma}, \cdots, \aleph_{\kappa+\omega}, \cdots, \aleph_{\kappa+\gamma}, \cdots \quad (\gamma < \zeta),$$

where ζ is a Limes-number and κ any ordinal number. Then we have

$$\prod_{\gamma<\zeta} \aleph_{\kappa+\gamma} > \sum_{\gamma<\zeta} \aleph_{\kappa+\gamma} = \aleph_{\kappa+\zeta},$$

and, supposing $\aleph_\xi$ to be the cardinal number of an aggregate which can be well-ordered in type ζ, and remarking that

$$\aleph_{\kappa+\zeta}^{\aleph_\xi} \geq \prod_{\gamma<\zeta} \aleph_{\kappa+\gamma},$$

---

* If we admit the multiplicative axiom, every $\mathfrak{m}$ is an Aleph, and the above condition is fulfilled if the product is determined by $\gamma < \zeta$, where ζ is a Limes-number. Then, also, there is no infinite series of $\mathfrak{m}$'s, say $\{\mathfrak{m}_\nu\}$, such that $\mathfrak{m}_{\nu+1} < \mathfrak{m}_\nu$; but it may be of some interest to see that these properties of the $\mathfrak{m}$'s are *not* used in the text.

we get the theorem that

$$\aleph_{\kappa+\zeta}^{\aleph_\xi} > \aleph_{\kappa+\zeta}, \tag{5}$$

where ζ and ξ are specified as above; and this includes König's theorem ($\zeta = \omega$, $\xi = 0$) as a particular case.

From (5) we conclude in just the way that König did in his special case, that, since

$$(2^{\aleph_\xi})^{\aleph_\xi} = 2^{\aleph_\xi \cdot \aleph_\xi} = 2^{\aleph_\xi},$$

we cannot have

$$2^{\aleph_\xi} = \aleph_{\kappa+\zeta},$$

where ζ is any Limes-number of the $(2+\xi)^{\text{th}}$ number-class.

August 10th, 1907.

[*Note added September 30th*, 1907. The above paper was sent to the editors of the *Mathematische Annalen*, and was by them referred to Prof. Zermelo for a report on it. Zermelo stated that my theorems were cases of a still further generalisation of König's theorem given by him in No. 33 of a paper "Grundlagen der Mengenlehre," which is shortly to appear in the *Mathematische Annalen*, and consequently my paper was not printed there. However, I cannot think this paper superfluous, since the method of generalisation of König's theorem does not seem to me wholly without interest, both in itself, and as an example of how practically the same result (for, as I will show, my theorems, when we drop the restriction of the class of the $\mathfrak{m}$'s being the same for both the Π and the Σ, which is essential in König's theorem, merge into Zermelo's theorem) was obtained by two people independently.

Zermelo's theorem is: If, for every γ in the sum and product considered,

$$\mathfrak{m}_\gamma{}' > \mathfrak{m}_\gamma,$$

then

$$\prod_\gamma \mathfrak{m}_\gamma{}' > \sum_\gamma \mathfrak{m}_\gamma.$$

It is not assumed that the class of $\mathfrak{m}$'s can be well-ordered, nor that the $\mathfrak{m}$'s are transfinite.

I proved the two theorems:

(1) If every M_γ is finite, then (§ III.)

$$\prod_\gamma (\mathfrak{m}_\gamma + 1) > \sum_\gamma \mathfrak{m}_\gamma.$$

(2) If every M_γ is transfinite, and if to each member M_γ of the class of all the M's considered corresponds a member M_ξ of the same class such that

$$\mathfrak{m}_\xi > \mathfrak{m}_\gamma,$$

then

$$\prod_\gamma \mathfrak{m}_\gamma > \sum_\gamma \mathfrak{m}_\gamma.$$

Now, in my generalisation of König's theorem, I kept the restriction that the $\mathfrak{m}$'s under the $\prod$ should always be the *same* class as those under the Σ; but, if we drop this restriction, and combine (2) with (1), we get Zermelo's generalisation.

It should be remarked that I also did not require (§ IV.) a well-ordering of the M_γ's; but I mentioned that, if we admit the multiplicative axiom, in general (a case of which we must when we speak of a product of an infinity of cardinal numbers), every $\mathfrak{m}$ is an Aleph, and the factors and summands can form well-ordered series. Thus it is easy to see that much (if not all) of the apparent generality of a theorem on infinite products gained from not assuming that an aggregate can be well-ordered, is illusory.]

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END OF VOL. XXXIX.

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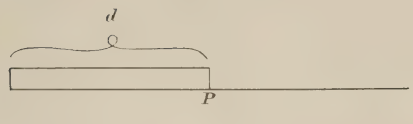

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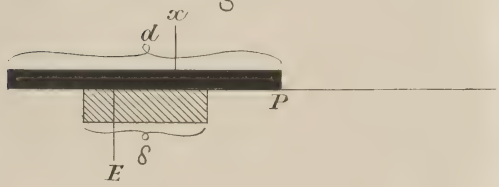

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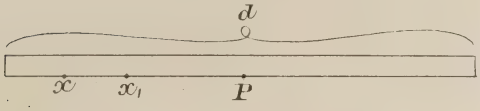

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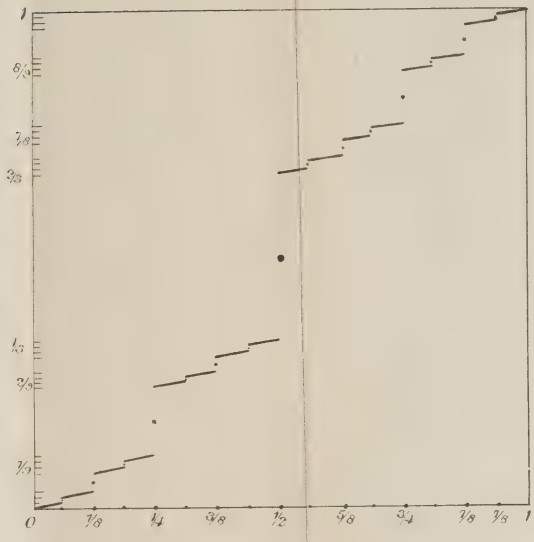

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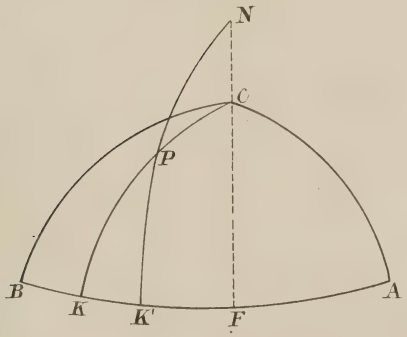

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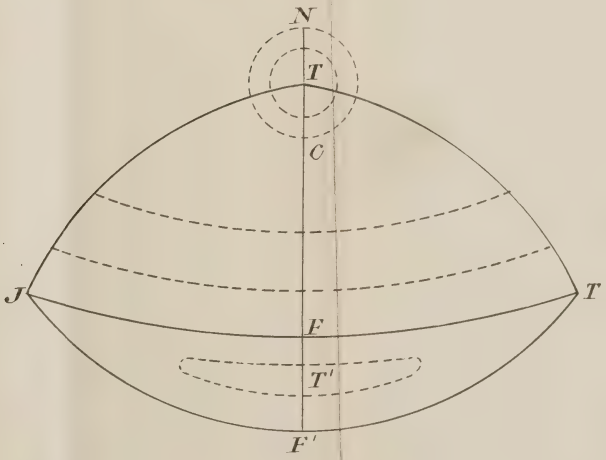

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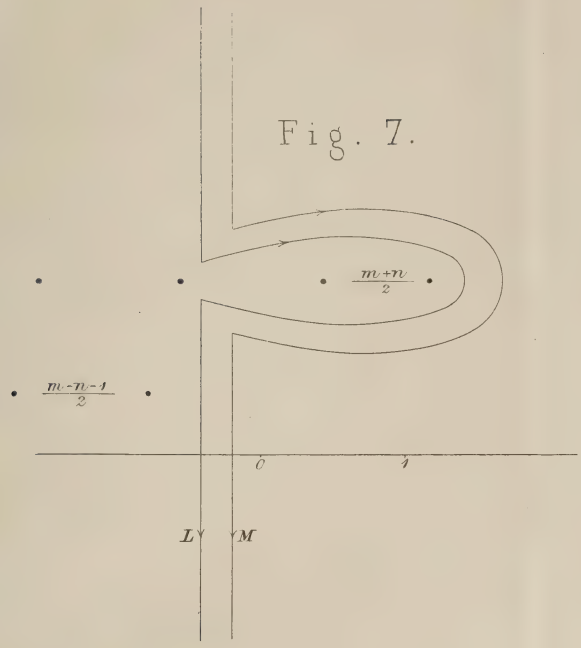

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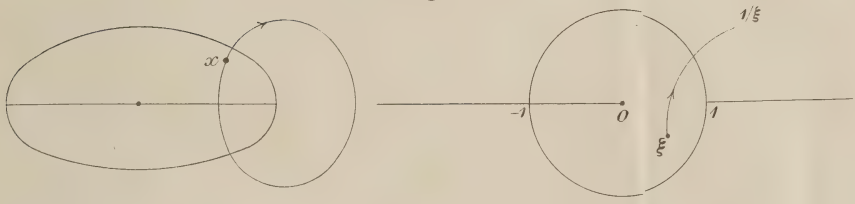

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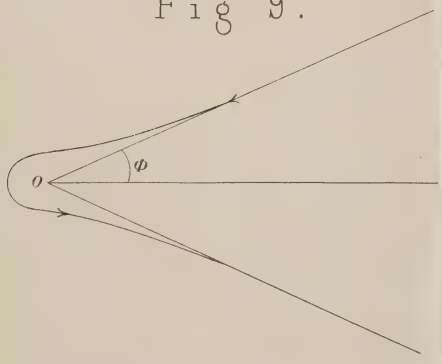

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